Isotropic Finsler Spaces
Classification and Examples

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Finsler Metrics: $F : TM \to R$

① $F(x, y) \geq 0$. "=" iff $y = 0$

② $F(x, \lambda y) = \lambda F(x, y) \quad \lambda > 0$

③ $g_{ij}(x,y) := \frac{1}{2} [F^2]_{y^i y^j}(x,y) > 0$

Geodesics: $\sigma(t)$

$$\frac{d^2\sigma^i}{dt^2} + 2G^i(\sigma, \frac{d\sigma}{dt}) = 0$$

$$G^i(x,y) := \frac{g^{il}}{4}\left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}$$

$TM_0 := TM \setminus \{0\}$

(x^i, y^i) standard local coordinate system

Let $\omega^i := dx^i \qquad \omega^{n+i} := dy^i + \frac{\partial G^i}{\partial y^j} dx^j$

Theorem (Chern) There exist a unique set of 1-forms $\omega_j{}^i = \Gamma^i_{jk}(x,y)\, dx^k$ such that

$$d\omega^i = \omega^j \wedge \omega_j{}^i$$

$$dg_{ij} = g_{kj}\, \omega_i{}^k + g_{ik}\, \omega_j{}^k + 2 C_{ijk}\, \omega^{n+k}$$

where

$$C_{ijk} = \frac{1}{4} [F^2]_{y^i y^j y^k} \qquad \text{(cartan torsion)}$$

$$\Rightarrow \omega^{n+i} = dy^i + y^j \omega_j{}^i \qquad \left(\frac{\partial G^i}{\partial y_j} = \Gamma^i_{jk} y^k \right)$$

Curvature :

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j{}^i$$

$$= \underbrace{\tfrac{1}{2} R^i{}_{kl}}_{\text{Riemann}}\, \omega^k \wedge \omega^l - \underbrace{L^i{}_{kl}}_{\text{Landsberg}}\, \omega^k \wedge \omega^{n+l}$$

$$R^i{}_k := R^i{}_{kl}\, y^l$$

$$R_{jk} := g_{ij} R^i{}_k \qquad L_{ijk} := g_{il} L^l{}_{jk}$$

Fact: $R_{jk} = R_{kj}$

L_{ijk} symmetric

$$L_{ijk} = C_{ijk|m}\, y^m$$

$$I_i := g^{jk} C_{ijk} \qquad J_i := g^{jk} L_{ijk}$$

Fact: $J_i = I_{i|m}\, y^m$

Fact: Riemannian

$\Leftrightarrow$ $C_{ijk} = 0$

$\Leftrightarrow$ $I_i = 0$

Definition: F is Landsbergian if $L_{ijk} = 0$

weakly Landsbergian if $J_i = 0$

Question: $L_{ijk} = 0 \overset{?}{\Leftrightarrow} J_i = 0$

Buseman Volume Form

$$dV = \sigma(x)\, dx^1 \cdots dx^n$$

$$\sigma(x) := \frac{\mathrm{Vol}\,(\mathbb{B}^n(1))}{\mathrm{Vol}\left\{ (y^i) \in \mathbb{R}^n \;\middle|\; F\left(y^i \frac{\partial}{\partial x^i}\Big|_x\right) < 1 \right\}}$$

Distorsion:

$$\tau(x,y) := \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}$$

S-Curvature:

$$S(x,y) := \frac{d}{dt}\left[\tau\left(\sigma(t), \frac{d\sigma}{dt}(t)\right)\right]_{t=0}$$

$\sigma(t)$ the geodesic with $\sigma(0)=x$, $\frac{d\sigma}{dt}(0)=y$

Fact:

$$I_i = \frac{\partial \tau}{\partial y^i} = \frac{\partial}{\partial y^i}\left[\ln \sqrt{\det(g_{ij})} \right]$$

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma)$$

In a standard local coordinate system

$$L_{ijk} = -\frac{1}{2} F F_{y^s} \frac{\partial^3 G^s}{\partial y^i \partial y^j \partial y^k}$$

Definition: F is a Berwald metric if

$$G^i(x,y) = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$$

Fact: If F is Berwaldian, then

① Landsbergian ($L_{ijk} = 0$)

② $S \equiv 0$ (S. 1996)

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

Definition: F is isotropic (of scalar curvature)

$$R_{ij} = K F^2 \left\{ g_{ij} - \frac{g_{is} y^s g_{jt} y^t}{F^2} \right\}$$

$K = K(x, y)$

Schur Lemma: If F is isotropic with $K = K(x)$, then

$K =$ constant $\quad (n \geq 3)$

Theorem/ Example: If F is locally projectively flat then F is isotropic. (Berwald)

If in addition, F is Riemannian then F is of constant sectional curvature (Beltrami)

Trivial Theorem: $F = \alpha + \beta$ is locally projectively flat

$\Leftrightarrow$ α is of constant sectional curvature

β is closed

Projectively flat

$\Leftrightarrow$ $G^i = P y^i$ (Berwald)

$\Leftrightarrow$ $F_{x^k y^l} y^k = F_{x^l}$ (Hamel, 1903 / Rapcsak, 1961)

$\left(P = \dfrac{F_{x^k} y^k}{2F} \right)$

F is projective on $\Omega \subset \mathbb{R}^n$ with flag curvature $K = K(x,y)$

$\Leftrightarrow$ $\exists\ P = P(x,y)$

$$\begin{cases} F_{x^k} = (PF)_{y^k} \\ P_{x^k} = PP_{y^k} - \dfrac{1}{3F}(KF^3)_{y^k} \end{cases}$$

can be solved easily for K = constant

Many interesting examples

$K = -1$

Minkowski norm

$$\Omega = \{ y \in \mathbb{R}^n \mid \phi(y) < 1 \}$$

Funk metric

$$\Theta(x,y) = \phi\big(y + \Theta(x,y)\, x \big)$$

① $F(x,y) = \frac{1}{2}\big(\Theta(x,y) + \Theta(x,-y) \big)$

(Hilbert metric)

② $\delta < 1$

$$F(x,y) = \frac{1}{2}\big(\Theta(x,y) - \delta\, \Theta(\delta x, y) \big)$$

if $\Omega = \mathbb{B}^n$

$$F(x,y) = \frac{1}{2}\left\{ \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2} - \delta\, \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)\delta^2} + \delta\langle x,y\rangle}{1-\delta^2|x|^2} \right\}$$

③ $F(x,y) = \frac{1}{2}\left\{ \Theta(x,y) + \frac{\langle a,y\rangle}{1+\langle a,x\rangle} \right\}$

if $\Omega = \mathbb{B}^n$

$$F(x,y) = \frac{1}{2}\left\{ \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle} \right\}$$

$K = 0$

① $$F(x,y) = \left\{ (1+\langle a,x\rangle)\Theta(x,y) + \langle a,y\rangle \right\}\left\{ 1+\Theta_{y^k}(x,y)x^k \right\}$$

if $\Omega = \mathbb{B}^n$

$$F(x,y) = \left\{ 1+\langle a,x\rangle + \frac{(1-|x|^2)\langle a,y\rangle}{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle} \right\} \times$$
$$\frac{\left(\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}$$

② $$F(x,y) = \frac{\phi\left((1+\langle a,x\rangle)y - \langle a,y\rangle x\right)}{(1+\langle a,x\rangle)^2}$$

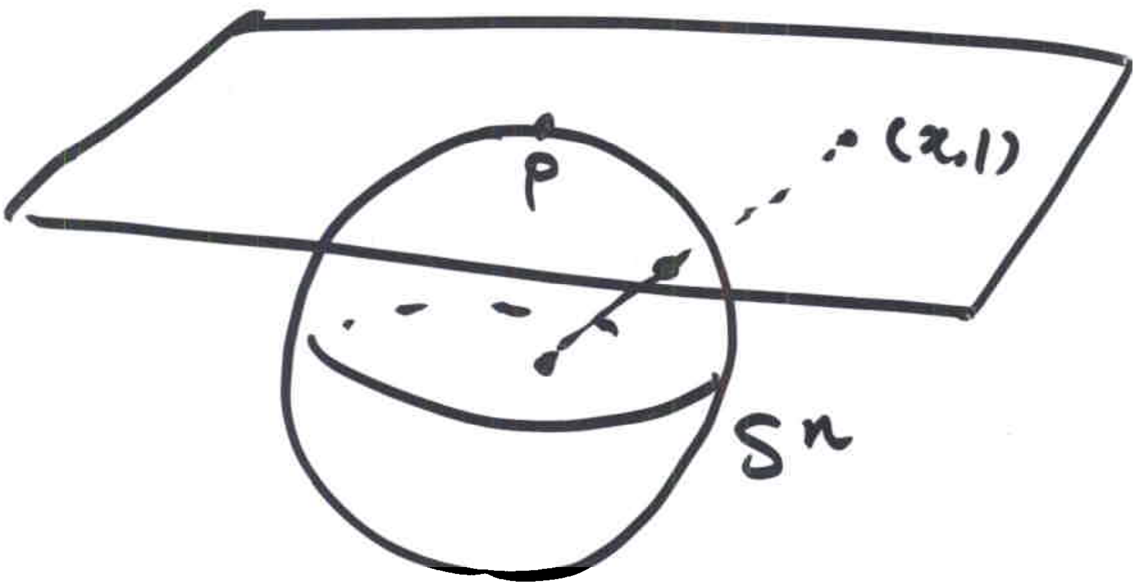
(locally Minkowskian)

$k = 1$

$|\alpha| < \pi/2$

$$H = H(x,y) = P(x,y) + i\,F(x,y)$$

$$H = \frac{-\langle x,y\rangle + i\sqrt{(e^{2i\alpha}+|x|^2)|y|^2 - \langle x,y\rangle^2}}{e^{2i\alpha}+|x|^2}$$



The pull-back Bryant metrics on $\mathbb{R}^n \cong T_p S^n$

The S-curvature

$$S(x,y) = \frac{d}{dt}\left[\tau\left(\sigma(t), \frac{d\sigma}{dt}(t)\right)\right]_{t=0}$$

1) Volume Comparison Theorem (1996)

2) Funk metric $\Theta(x,y)$ on $\Omega \subset \mathbb{R}^n$

$$S(x,y) = \frac{n+1}{2} F(x,y)$$

3) $$F(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2} \pm \frac{\langle a,y\rangle}{1+\langle a,x\rangle}$$

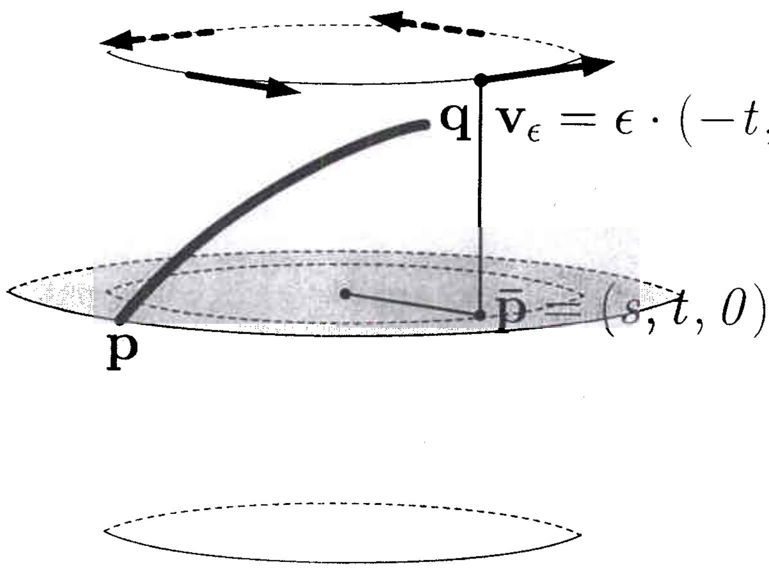
$$\begin{cases} K = -\frac{1}{4} \\ S = \pm\frac{n+1}{2} F \end{cases}$$

4) All known non-locally projectively flat Randers metrics $F = \alpha + \beta$ of constant flag curvature

$S = (n+1)cF$, $c = \text{constant}$

Bao-Shen, Shen, Bao-Robles

$\Phi(x, y) = \sqrt{g(y, y)}$ is the standard Riemannian metric on S^2.

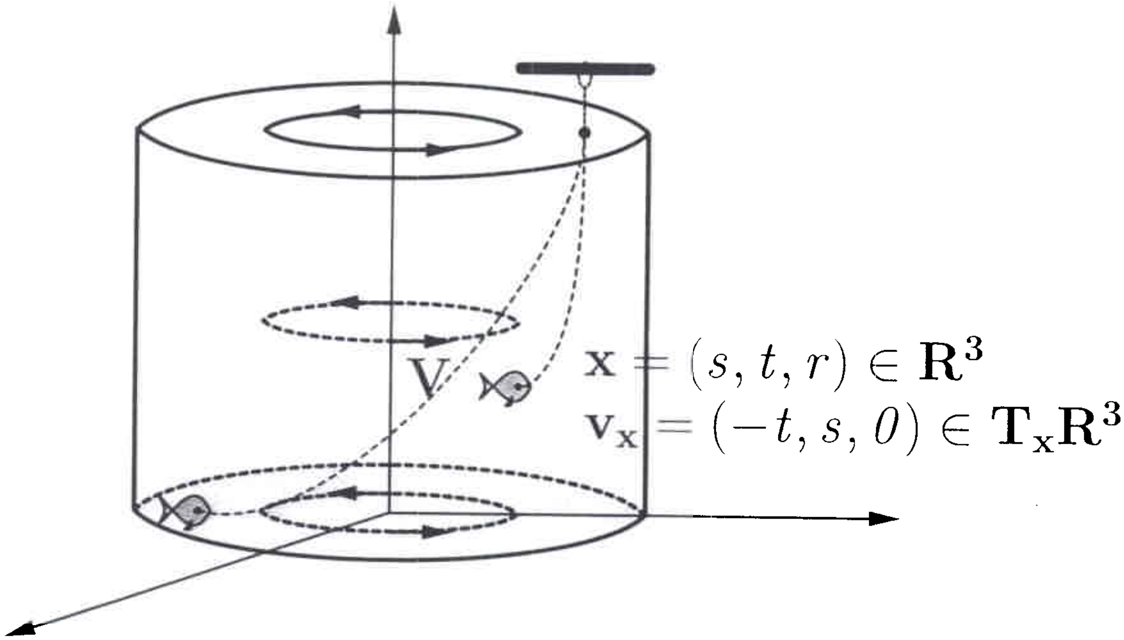


Shortest time paths = Geodesics of F:

$$F(x, y) = \frac{\sqrt{g(y, y)(1 - g(v_\epsilon, v_\epsilon)) + g(v_\epsilon, y)^2} - g(v_\epsilon, y)}{1 - g(v_\epsilon, v_\epsilon)}.$$

- **Flag curvature K = 1.**
- $S = 0$

$\Phi(\mathbf{x}, \mathbf{y}) = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$ is the Euclidean metric on $\mathbf{R}^3$.



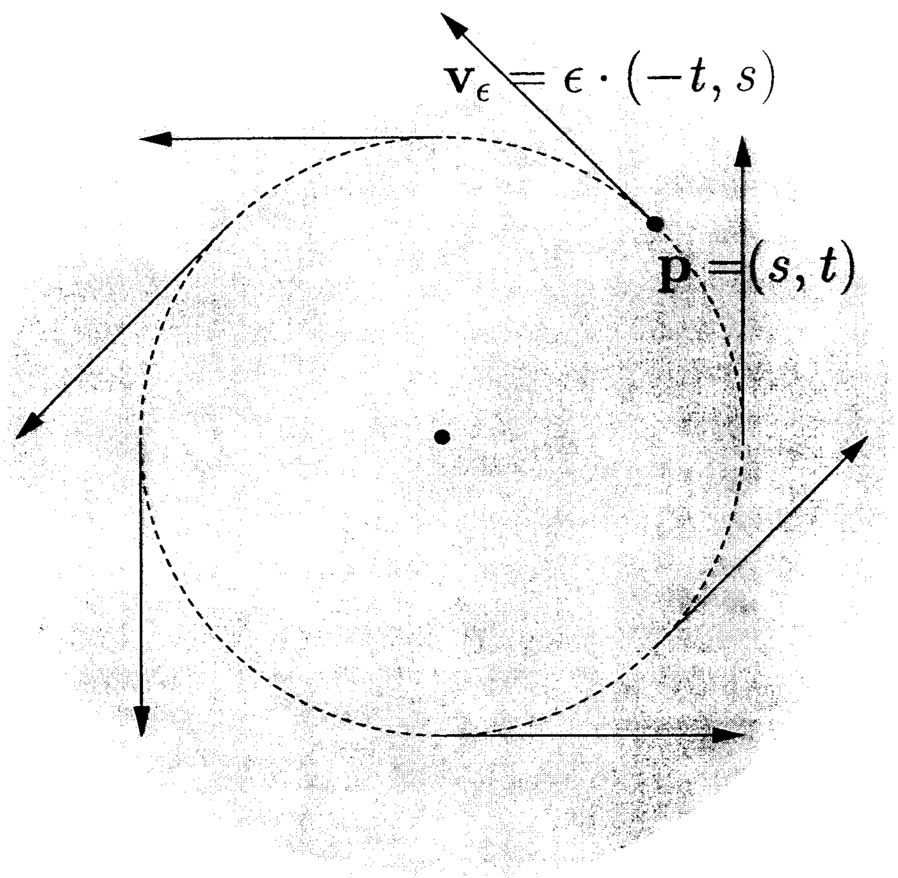
Shortest Time Paths = Geodesics of F:

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \frac{\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle (1 - \langle \mathbf{v}, \mathbf{v} \rangle) + \langle \mathbf{v}, \mathbf{y} \rangle^2} - \langle \mathbf{v}, \mathbf{y} \rangle}{1 - \langle \mathbf{v}, \mathbf{v} \rangle}.$$

- **Flag curvature $\mathbf{K} = \mathbf{0}$.**

- $S = 0$

$\Phi(\mathbf{x}, \mathbf{y}) = \sqrt{\mathbf{g}(\mathbf{y}, \mathbf{y})}$ **is the Klein metric on** B^2.



Shortest time paths = Geodesics of F:

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \frac{\sqrt{\mathbf{g}(\mathbf{y}, \mathbf{y})(1 - \mathbf{g}(\mathbf{v}_\epsilon, \mathbf{v}_\epsilon)) + \mathbf{g}(\mathbf{v}_\epsilon, \mathbf{y})^2} - \mathbf{g}(\mathbf{v}_\epsilon, \mathbf{y})}{1 - \mathbf{g}(\mathbf{v}_\epsilon, \mathbf{v}_\epsilon)}.$$

- **Flag curvature** $\mathbf{K} = -1$.
- $S = 0$

5) Bao-Robles (2001)

if $F = \alpha + \beta$ satisfies

$Ric = (n-1)K(x) F^2$

then $S = (n+1) c F$ $\quad$ c = constant

6) Kim-Yim

(S^n, F) if F reversible

$K = 1$

$S = 0$

then F is Riemannian

Question: F on S^n, reversible, $K=1$

is F Riemannian?

7) Shen (2002)

(M, F) compact

$K = K(P, y) \leq 0$ (flag curvature)

$S = (n+1) c F$ $\quad$ c = constant

then F is weakly Landsbergian $(J=0)$

F is Riemannian at $x \in M$

where $K < 0$

Example

Let (M_1, α_1), (M_2, α_2) be compact Riemannian manifolds of $K \le 0$

$\phi : R_+ \times R_+ \to R_+$ $\quad \phi(\lambda u, \lambda v) = \lambda \phi(u,v)$, $\lambda > 0$

$$F(x,y) = \phi\left(\alpha_1(x_1, y_1), \ \alpha_2(x_2, y_2)\right)$$

$x = (x_1, x_2) \in M := M_1 \times M_2$

$y = y_1 \oplus y_2 \in T_x M = T_{x_1} M_1 \oplus T_{x_2} M_2$

1) F is a Berwald metric (following Szabo)
$\Rightarrow L = 0 \Rightarrow J = 0$

2) Flag curvature $K \le 0$ $\quad (K \not\equiv 0)$

$\phi(u,v) = u h(v/u)$ $\qquad h = h(s)$

F is a Finsler metric $\Leftrightarrow$

$h' > 0, \quad h - s h' > 0$

$h^3 h'' - [(h - sh')h' - s h h'']^2 > 0$

Projective flat + $S = (n+1)c(x)F$

$$\Downarrow$$

$$K - \frac{3c_{x^k}y^k}{F} = \sigma(x)$$

$\xrightarrow{\sigma + c^2 \neq 0}$

$F = \alpha + \beta$

Randers metric

$\downarrow$

$F = \alpha + \beta$
projectively flat
$S = (n+1)c(x)F$
Chen – Shen
May, 2002

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$\xrightarrow{\sigma + c^2 = 0}$

$\xrightarrow{c=0}$ Locally Minkowski

$\xrightarrow{c \neq 0}$

c = constant

$\downarrow$

$F = \frac{1}{2c}(\Theta - \beta)$

Θ — Funk metric

$K = -c^2 \leq 0$

$\beta = -\frac{\langle a, y \rangle}{1 + \langle a, x \rangle}$

$F=\alpha+\beta$ Chen - Shen (2002)

Projectively flat $+$ $S=(n+1)c(x)F$

$\Updownarrow$

α — constant curvature $=\mu$

β — closed

$S=(n+1)c(x)F$

$\downarrow$

$$c_{x^k}y^k=-\frac{1}{2}(\mu+4c^2)\beta$$

$$K=\frac{3c_{x^k}y^k}{F}+\mu+3c^2$$

$\mu+4c^2=0$:

$$\boxed{\begin{array}{l} c=\text{constant} \\ K=-c^2 \end{array}}$$

$\downarrow$

$$\boxed{\begin{array}{l} F=\alpha+\beta \\ \text{projectively flat} \\ \text{with constant} \\ \text{flag curvature} \\ \text{Shen (2001)} \end{array}}$$

$\mu+4c^2\neq 0$:

$$\beta=\frac{-2c_{x^k}y^k}{\mu+4c^2}$$

$$\boxed{\begin{array}{l} \text{three cases} \\ \mu=-1,\ \mu=0, \\ \mu=1 \\ \text{Nex page} \end{array}}$$

① $\mu = 0$

$F = \alpha + \beta$ projectively flat

$$\alpha = |y|, \quad \beta = \frac{-2c_{x^k}y^k}{4c^2}$$

$$c = \frac{1}{|\sqrt{1+|x|^2} + \langle a, x\rangle + \lambda|} \qquad \lambda > 0$$

$$K = 3\left\{\frac{c_{x^k}y^k}{\alpha+\beta} + c^2\right\} + 0$$

② $\mu = -1$

$F = \alpha + \beta$ projectively flat

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2}, \quad \beta = \frac{-2c_{x^k}y^k}{-1+4c^2}$$

$$c = -\frac{\langle a, x\rangle + \lambda}{2\sqrt{(\lambda + \langle a, x\rangle)^2 - (1-|x|^2)}}$$

$$K = 3\left\{\frac{c_{x^k}y^k}{\alpha+\beta} + c^2\right\} - 1$$

③ $\mu = 1$

$F = \alpha + \beta$ projectively flat

$$\alpha = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - (x,y)^2)}}{1+|x|^2}, \quad \beta = \frac{-2c_{x^k}y^k}{1+4c^2}$$

$$c = \frac{\langle a,x\rangle + \varepsilon}{2\sqrt{1+|x|^2 - (\langle a,x\rangle + \varepsilon)^2}} \qquad |\varepsilon| < 1$$

$$K = 3\left\{ \frac{c_{x^k}y^k}{\alpha+\beta} + c^2 \right\} + 1$$

<u>Theorem</u> (Shen, 2001)

$F = \alpha + \beta$ locally projectively flat with constant curvature

then $K = -c^2 \leq 0$

1) $c = 0$ locally Minkowskian

2) $c = \pm 1/2$

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2} \pm \frac{\langle a,y\rangle}{1+\langle a,x\rangle}$$

F isotropic $K = K(x,y)$

$S = (n+1)\,c(x)\,F$

<u>Theorem</u> (Chen - Mo - Shen, 2002)

$$K = \frac{3 c_{x^k} y^k}{F} + \sigma(x)$$

If in addition, $c = $ constant, then

$$K = \sigma(x)$$

$$= \text{constant} \quad (n \geq 3)$$

$n \geq 3$ F has constant ~~flag~~ S-curvature and isotropic

then F has constant flag curvature

(Mo)

F isotropic $\quad K = K(x,y)$

$$J = -c(x)\, I\, F$$

<u>Theorem</u> (Chen-Mo-Shen, 2002)

$$\frac{n+1}{3}\frac{\partial K}{\partial y^k} + \left(K + c^2 - \frac{c_{x^k} y^k}{F}\right)\frac{\partial \tau}{\partial y^k} = 0$$

If $c =$ constant, then

$$K = -c^2 \sigma(x)\, e^{-\frac{3}{n+1}\tau(x,y)}$$

Theorem (Mo-Shen, 2002) $(n \geq 3)$

(M, F) compact, isotropic

$$K = K(x,y) < 0$$

then $F = \alpha + \beta$

$\left\{ \begin{array}{l} \text{if } K = \text{constant} < 0 \\ \text{then } F = \alpha \quad \text{(Akbar-Zadeh)} \end{array} \right.$

Example: (M, α) compact Riemannian manifold with constant curvature $\mu < 0$, β is a closed 1-form.

$$F_\epsilon = \alpha + \epsilon\beta$$

For sufficiently small ε, F_ϵ is isotropic with flag curvature $K < 0$

Matsumoto torsion

$$M_{ijk} = C_{ijk} - \frac{1}{n+1}\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \}$$

$$I_i = g^{jk} C_{ijk}, \quad h_{ij} = g_{ij} - g_{ip} y^p g_{jq} y^q / F^2$$

Theorem (Matsumoto-Hōjō) $(n \geq 3)$

F is a Randers metric $\Leftrightarrow M_{ijk} = 0$

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if F is isotropic with $K = K(x,y)$

then

$$M_{ijk|p|q} y^p y^q + K F^2 M_{ijk} = 0$$

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if (M, F) compact, isotropic

$K = K(x,y) < 0$

then $M_{ijk} = 0 \Rightarrow F = \alpha + \beta$

$M''(t) + K(t) M(t) = 0.$ $\qquad K(t) < 0$

$\overline{}$

R

$\Rightarrow M = 0$