Second tangent bundle. Vertical derivative.

$$\begin{tikzcd} TTM \arrow[r, "Tp_M"] \arrow[d, "p_{TM}"'] & TM \arrow[d, "p_M"] \\ TM \arrow[r, "p_M"] & M \end{tikzcd}$$

, $f \in \mathcal{C}^1(TM;\mathbb{R})$

$w \in TM$, $Z \in T_w TM$

$$d_v f_w(Z) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[f\big(w + \varepsilon\, Tp_M(Z)\big) - f(w)\Big]$$

vertical derivative.

Homogeneous bundle = Bundle of tangent half Lines (often identified with UM).

$HM = \mathring{T}M / \mathbb{R}^{+,*}$, $\mathring{T}_x M = T_x M - \{0\}$.

$\mathring{T}M \xrightarrow{r} HM$.

- If $F : TM \to \mathbb{R}$ is a $\mathcal{C}^2$ Finsler metric there exists a unique one-form A on HM such that

$$r^* A = d_v F.$$

lift of curves $c : I \to M$, $\tilde{c} : I \to HM$, $\tilde{c} = r(\dot{c})$

Second order differential equation

Def. A second order differential equation on M is a section $X : HM \to THM$ such that

$$\begin{tikzcd} THM \arrow[rr, "T\pi"] & & \mathring{T}M \arrow[r, "r"] \arrow[dl] & HM \\ HM \arrow[u, "X"] \arrow[r, "\pi"] & M & & \end{tikzcd}$$

- X is nowhere vertical.
- $r \circ T\pi \circ X = Id(HM)$.

Geometrically

A second order differential equation on M is a vector field on HM whose integral curves are lifted curves of the base M.

(2) Geodesics Equation

Thm. 1. For a ~~smooth~~ strongly convex Finsler metric F over M. the one form A is a contact form. that is $A \wedge dA^{n-1}$ is a volume form on ($HM \simeq UM$ (see later).

($n = \dim M$).

Thm. 2. The generator of the geodesic flow.

i) Is a second order differential equation X on M.

ii) This vector field is the Reeb field of the contact form.

$dA(X, \ ) = 0$

$A(X) = 1$.

From Now Finsler geometry inherits of two kinds of geometry.

Contact geometry + Geometry of second order differential equations.

(3)

If F is regular and strictly convex, then A is a contact form and X is the associated Reeb field. Let VHM be the vertical bundle over HM. The 1-form A is semi-basic, that is,

$$A/VHM = 0\ . \tag{4.3}$$

Second order differential equations

Let r be the map that sends each non zero vector into its corresponding half-line and $T\sigma$ the tangent map to the projection σ.

Definition 1. A second order differential equation on M is a vector field, X without singularities on HM such that

$$r \circ T\sigma \circ X = \text{Identity of } HM; \tag{4.4}$$

or equivalently: *the integral curves of X are the canonical lifted curves of M.*

Proposition 3. *The geodesic vector field X of a Finsler metric is a second order differential equation.*

Remark : The same construction can be done for the projectivized bundle PTM.

Dynamical approach

(see [22]) : At this step for any C^2 second order differential equation X we can construct a dynamical derivative and a Jacobi endomorphism without making any use of Finsler geometry.
(For instance, X will not be supposed to have an invariant volume form.)
The key lemma is the following.

Lemma 13. *Let $X : HM \to THM$ be a second order differential equation, Y_1 and Y_2 two vertical vector fields on HM. If at one point $z \in HM$, there exist two real numbers a, b, such that*

$$aX(z) + Y_1(z) + b[X, Y_2](z) = 0\ ,$$

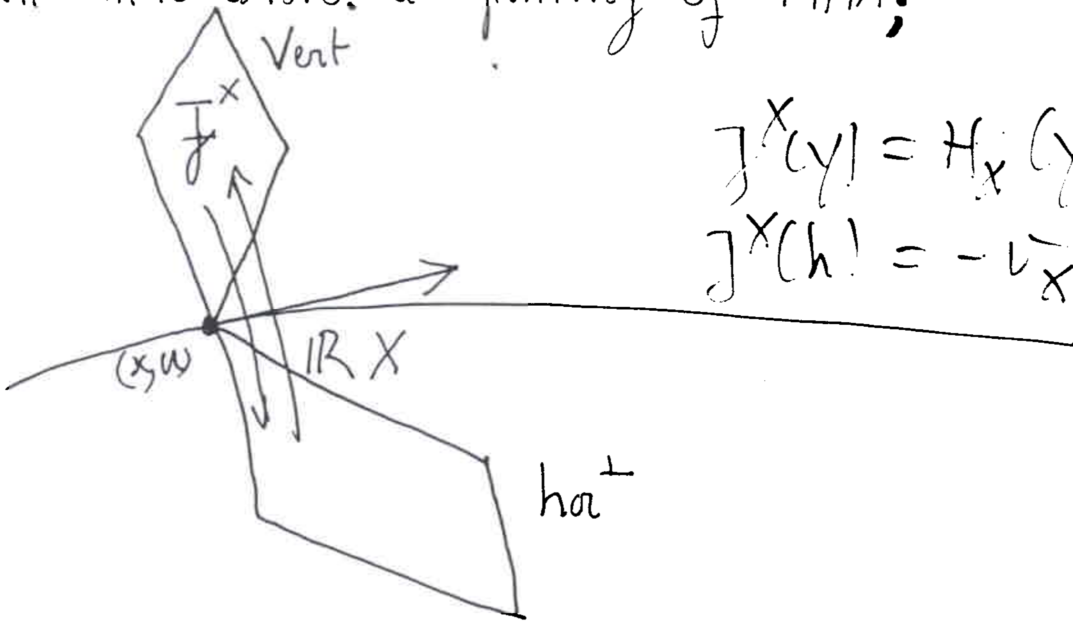
then $bY_2(z) = 0$ and $a = 0$.

Furthermore if $Y_2(z)$ is not zero, then $b = 0$ and $Y_1(z) = 0$.
A direct application is :

④ **Geometry of a second order differential Equation.**

Assume $X : HM \to THM$ given (No Finsler metric a priori)

Then there exists: a splitting of THM;



$$\mathcal{J}^X(Y) = H_X(Y)$$
$$\mathcal{J}^X(h) = -V_X(h)$$

- $THM = \mathbb{R}X \oplus hor^\perp \oplus Vert$;
- A pseudo complex structure $\mathcal{J}^X$ on $Vert \oplus hor^\perp$;
- A dynamical derivative $\mathcal{D}^X$, that is a first order differential operator acting on the sections of the bundle THM such that

  $\forall\ \xi \in \mathcal{C}^{(1)} HM,\ \forall f \in \mathcal{C}^1(HM, \mathbb{R})$.

  $$\mathcal{D}^X(f.\xi) = f\,\mathcal{D}^X(\xi) + \xi \,.\, L_X(f)$$ ($L_X f$ is the Lie derivative along X)

- The splitting and the pseudo complex structure $\mathcal{J}^X$ are parallel for $\mathcal{D}^X$! (That is for the corresponding parallel transport).
  - $\mathcal{D}^X\, Vert \subset Vert$, $\mathcal{D}^X X = 0$, $\mathcal{D}^X hor^\perp \subset hor^\perp$
  - $\mathcal{D}^X \mathcal{J}^X = 0$.

Time Change

Prop. Let X a second order differential équation. $m \in \mathcal{C}^\infty(HM, \mathbb{R}^{*,+})$

then

$$\mathcal{D}_{mX} = m\,\mathcal{D}_X + \frac{1}{2} L_X m \cdot Id.$$

$$R^{mX} = m^2 R^X + \left\{ \frac{1}{2} m L_X^2 m - \frac{1}{4} (L_X m)^2 \right\} Id.$$

Back to Finsler Geometry.

Prop. If F is a smooth Finsler metric on M. There exists a natural riemannian metric $\tilde{g}$ on HM such that.

i) the splitting $\mathbb{R}X \oplus h\alpha^\perp \oplus V$ is orthogonal.

ii) $\tilde{g}(X,X) = 1$.

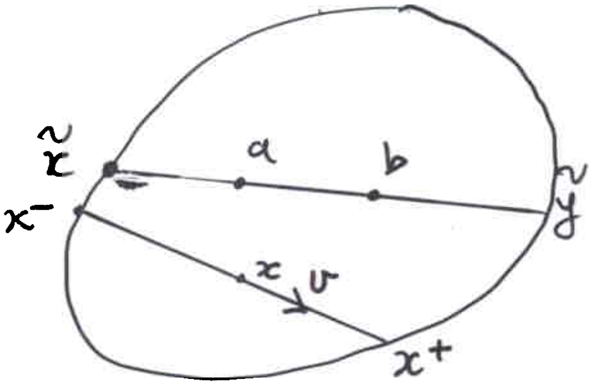
iii) $\tilde{g}_u(h_1, h_2) = \tilde{g}_u(J^X(h_1), J^X(h_2))$.

iv) $\tilde{g}_u(Y_1, Y_2) = dA_u([X, \tilde{Y}_1], Y_2)$

(where Y_1, Y_2 are vertical vectors | $\tilde{Y}_1$ is a $\mathcal{C}^1$ vertical extension).

Prop The metric g is $\mathcal{D}^X$ parallel.
The Jacobi Endomorphism is symetric.

(6) Examples
Hilbert Geometry



Let $\mathcal{C}$ be an open bounded convex domain in $\mathbb{R}^n$.

$$d^h(a,b) = \frac{1}{2} \ln [a,b,\tilde{x},\tilde{y}]$$
$$= \frac{1}{2} \ln \left(\frac{d^e(a,\tilde{y}) \cdot d^e(b,\tilde{x})}{d^e(a,\tilde{x}) \cdot d^e(b,\tilde{y})} \right)$$

(d^e is the euclidian distance)

If $\mathcal{C}$ is strictly convex (or has only one segment) then geodesics are straight

The euclidian flow is given by (on $H\mathcal{C}$).

$$\varphi_t^E(x,[v]) = \left(x + t\frac{v}{\|v\|}, [v] \right), \quad X^E(x,[v]) = \frac{d}{dt} \varphi^E_{t}(x,[v]) \Big|_{t=0}$$

The Hilbert flow generator satisfies thus

$$X^H = m.X^E \quad \text{with} \quad m(x,[v]) = \frac{2}{\left(\frac{1}{d^e(x,x^+)} + \frac{1}{d^e(x,x^-)} \right)} \quad X^E$$

$$L_{X^E} m = \frac{d}{dt} m\left(x + \frac{v}{\|v\|} t, [v] \right) = \frac{d}{dt} m(x_t,[v])$$
$$= 2\frac{d}{dt} \left(\frac{d(x_t,x^+)\, d(x_t,x^-)}{d(x_t,x^+) + d(x_t,x^-)} \right) = \frac{2}{\lambda_t + \mu_t} (-\mu_0 + \lambda_0)$$

$$\mu_t = d(x_t, x^+), \quad \lambda_t = d(x_t, x^-)$$

$$L^2_{X^E} m = -\frac{4}{\lambda_0 + \mu_0}$$

$$R^X = m^2\, 0 + \frac{1}{2}\, 2\left(\frac{\lambda_0 \mu_0}{\lambda_0 + \mu_0} \right) \cdot \frac{-4}{(\lambda_0 + \mu_0)} - \frac{1}{4} \left(\frac{2}{\lambda_0 + \mu_0} \right)^2 (-\mu_0 + \lambda_0)^2$$

$$\left(m^2 R + \frac{1}{2} m L^2_{X^E} m - \frac{1}{4} (L_{X^E} m)^2 \right.$$

$$R^X = -1$$

(7)

Hilbert Geometry. some interesting properties.

- The Hilbert Finsler metric is reversible.
- Geodesics are unique iff $\mathcal{C}$ is strictly convex
- Then F is projectively flat.

If $\partial \mathcal{C}$ is smooth (C^2)

- The metric is parallel furthermore.
  the Jacobi endomorphism is negative constant

$$R = -\mathrm{Id}.$$

(This was proven by Funk for surfaces then by ...)

Thm (Egloff. Busemann) (Smooth, strict convex case).

- A Hilbert geometry is locally symetric iff.
  $\mathcal{C}$ is the interior of an ellipsoid

(Then it is the Klein model of real hyperbolic geometry).

Thm (P. Funk, L. Berwald, 1928)

let M be a simply connected manifold, F a reversible Finsler metric which is furthermore.

- projectively flat, complete.
- with constant negative curvature.

Then it is a Hilbert geometry.

Terminology.

A Finsler metric is said to be <u>reversible</u> iff
$F(v) = F(-v)$

reason. The geodesic flow is then conjugate to its inverse by the antipody $\sigma(v) = -v$.

In 1926 E. Cartan introduced the Riemannian Locally symetric spaces as the one whose curvature tensor is parallel.

<u>Riemannian geometry</u>.

parallel curvature $\quad\iff\quad$ Locally symetric

$\nabla R = 0 \quad\iff\quad \forall x_0 \in M$

the geodesic symetric s_{x_0} is a local isometry.

Finsler geometry.

<u>"parallel curvature"</u> $\quad$ <u>Locally symetric</u>

$\nabla_X R = 0 \quad \Longleftarrow$

$\not\Longrightarrow$ (see Hilbert geometry)

parallel Finsler space $\quad$ Locally symetric Finsler space.

A Finsler metric is <u>locally projectively flat</u> if any point admits a chart such that the geodesics are

- Negative flag curvature -

We can now state the main rigidity theorem. The results of this part have been previously announced in [25]. Here we give a complete proof.

Theorem 1. *A compact Finsler space with parallel negative curvature is isometric to a Riemannian locally symmetric negatively curved space.*

Combining Theorem 1 with Proposition1 immediately implies the following.

Corollary 1. *A locally symmetric compact Finsler space with negative curvature is isometric to a negatively curved Riemannian locally symmetric space.*

Theorem 1 contains, as a particular case compact manifolds with constant curvature, for which this result was known by a theorem of Akbar Zadeh [37]. (The proof he gives is very different from ours.)

Let us remark that there exist compact continuous locally symmetric Finsler spaces (see the nice examples by Verovic ([36]) of higher rank which are not isometric to Riemannian models).

This theorem also brings a complete answer to a result that I obtained in [23] and which extends to the Finsler setting a result of R. Osserman and P. Sarnak [33]. This result provides an estimate from below for the metric entropy of a closed negatively curved Finsler manifold. I can now state it in a complete form which includes the analysis of the equality case.

Theorem 2. *The Liouville entropy h_μ of the geodesic flow of a closed reversible Finsler space (M,F) with negative Jacobi endomorphism, R, satisfies the following inequality*

$$h_\mu \geq \int_{HM} Tr((-R)^{1/2}).\, d\mu \,, \tag{1.1}$$

with equality if and only if (M,F) is isometric to a Riemannian locally symmetric space of negative curvature.

Another nice direct application is a new proof of a result of Benzecri ([7]) .

Corollary 2. *Let C be a convex bounded set in $\mathbb{R}^n$ with smooth strictly positive boundary (positive second-fundamental form) ∂C. Assume furthermore that*
i) there exists a subgroup $\Gamma \in PGl(\mathbb{R}^n)$ which preserves C .
ii) $\Gamma \setminus C$ is a closed manifold.
Then C is the interior of an ellipsoid.

For these good smooth Hilbert geometries, Edith Socié obtains a complete result [35] which goes beyond Corollary 2.

Theorem (E. Socié). *Let C be a convex bounded set in $\mathbb{R}^n$ with C^2 strictly positive boundary. Then the isometry group of the corresponding Hilbert geometry is compact except if ∂C is an ellipsoid.*

This proves in particular that the only non-trivial quotients of a Hilbert geometry which are manifolds without boundary are Riemannian.

- *Positive flag curvature* -

In this part, I will only consider Finsler metrics on the 2-sphere whose flag curvature is $+1$. The results that I announce here, are joint work with A. Reissman [21]. There are very good references on the subject. One is Robert Bryant's text in this issue and his other texts on the same topic for instance [10]. I also recommend the texts of Zhongmin Shen (in particular[34]) and the recent work of D.Bao and Z. Shen ([5]). What we have observed is that the dynamics of the geodesic flow is very simple. We obtain a result comparable in essence to the one of A. Weinstein for the Zoll metrics.

Theorem 3. *The geodesic flow on the homogeneous bundle of a reversible complete smooth Finsler structure of curvature $+1$ on the two-sphere is smoothly conjugate to the geodesic flow of the standard sphere.*

It is not clear whether this conjugacy result still holds true for non reversible metrics. The idea behind the proof of this theorem is (following Bryant in [10]) to transport all flow invariant structural objects of the Finsler metric to the space of geodesics Λ, which is a sphere. Then the geodesic field can be viewed as the vertical field of a Riemannian metric on Λ, and we conclude by using a conformal change of this metric.

The geometry of such a metric is far from being understood if we do not add complementary assumptions on the nature of the geodesics. For instance, if one assumes that the metric is locally projectively flat, then in the reversible case it is known for a long time by a work of Funk that the metric has to be Riemannian [26]. The general case (non reversible) is completely handled by R. Bryant in [11]. As previously said, in our case, the space Λ of oriented geodesics is a also a 2-sphere. We observe then that the fibers of the homogeneous bundle project on, Λ, to a family of curves, that we call Y-curves. They have the same incidence properties as geodesics of the Euclidean sphere (see Proposition 8 for a precise statement). Due to the reversibility, we obtain an abstract projective two-plane

(11)

An example of Finsler G-invariant metric in higher rank

$M = SL(n,\mathbb{R})/SO(n)$, $P_0 = SO(n)$

$G = PSL(n,\mathbb{R})$

$\mathfrak{g} = \{M \in M_n(\mathbb{R}),\ \mathrm{Tr}\, M = 0\}$

$\mathfrak{k} = \{M \in M_n(\mathbb{R}),\ {}^tM + M = 0\}$

$\mathfrak{p} = \{M \in M_n(\mathbb{R}),\ {}^tM = M,\ \mathrm{tr}\, M = 0\}$

$\mathfrak{a} = \{M \in M_n(\mathbb{R}),\ \text{diag matrix},\ \mathrm{tr}\, M = 0\}$

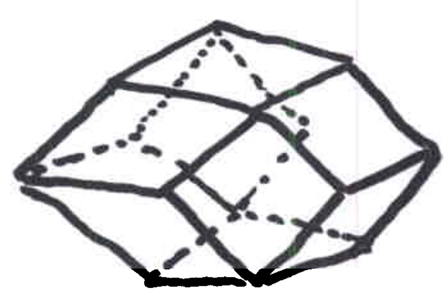
$$B = \{\overline{\mathrm{diag}(\lambda_1, \dots, \lambda_n)} \in \mathfrak{a} \quad |\lambda_i - \lambda_j| \leq 1\ \forall i,j\}$$

$n=2$

$\begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ $\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$n=3$

$n=4$



rhombic dodecahedron.

Symmetric Spaces of higher Rank

(M, g) g riemannian, $x_0 \in M$

G . connected component of identity in the isometry group.

K . Stabilizer of x_0 in G

$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition.

$\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra

$\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber.

$\Lambda_{\mathfrak{a}^+}$ the set of positive roots with respect to $\mathfrak{a}^+$

Th (Verovic) - Let F be a G-invariant Finsler metric on M then.

$$h_v(F) = \max_{H \in \overline{\mathfrak{a}^+ \cap B_F}} \Big(\sum_{\alpha \in \Lambda_{\mathfrak{a}^+}} m_\alpha \, \alpha(H) \Big)$$

(closure in $\mathfrak{a}$)

Dynamics and geometry.

Due to the contact form and the convexity, many theorems remains true in the Finsler setting.

- Finsler Gauss Lemma. (geodesics are locally minimizing).
- Hopf. Rinow theorem.
- Myers $Ric_u = Tr_u R^X$, $Ric \geq (n-1) r^2$ + complete $\Rightarrow diam(M,F) \leq \pi r$
- Synge $n = 2p$, compact, orientable, $R^X > 0 \Rightarrow$ simply connected
- Hadmard Cartan. $R^X \leq 0$ complet simply connected $\simeq \mathbb{R}^n$.
- In negative curvature the flow is an Anosov flow.

⋮

(no slide for this part)

Curvature and topology on complete Finsler Manifolds

- If $Ric_u > c > 0$ then $\pi_1(M)$ is finite. (Myers).
- If R^X is ~~[illegible]~~ negative symply connected then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$ (Hadamard Cartan).

Furthermore.

Thm (Egloff) The fundamental group of a compact Finsler manifold (M,F) with strictly negative curvature has exponential growth.

Thm (extension of a result of Cartan) (M, F) symply connected. Finsler manifold of non positive curvature. Then every compact group of isometries has a common fixed point.

Thm (Kreissmann) (M,F) compact $R < 0$. Then every abelian subgroup Γ_0 of $\pi_1(M)$ has a unique common axis and is infinite cyclic.

Remarks on parallel Spaces.

1) The projective equivalence problem

Theorem. Let (M,F) be a (C^3) parallel Finsler space.
If (M,F) is locally projectively equivalent to a locally symmetric riemannian space (S,g) with.

$$S \in \{ \mathbb{C}H_n, \mathbb{K}H_n, \mathbb{C}a_2 H, \mathbb{C}P_m, \mathbb{K}P_n, \mathbb{C}a_2 P \}$$

then F is riemannian and locally isometric to S. (up to a scalar factor)

Sketch of proof: let $\mathcal{D}, R$ be the dynamical derivative and curvature (Jacobi) of (S,g)

$\mathcal{D}', R'$ .... of (M,F)

$$R_* = m^2 R + \{ \underbrace{\tfrac{1}{2} m L_X m^2 - \tfrac{1}{4}(L_X m)^2}_{f} \} \, Id.$$

$$\mathcal{D}_* R_* = 0, \quad \mathcal{D} R = 0 \Rightarrow \qquad 2 m L_X m \, R + \dot{f} \, Id = 0$$

Here R and Id are independant then

$L_X m = 0$, but

$$A_* = m A + i_{u} dA$$

u: vertical vector field

A: riemannian symmetric space.

the geodesic equation $\Rightarrow$ $[X,[X,u]] = 0$, $dm = dA_{[X,u]}$

$\Rightarrow$ $[X,u]$ horizontal [redacted] and furthermore

$R(u) = 0$

Thus m is constant.