A Sphere Theorem for non-reversible Finsler metrics

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Question of Global (Riemannian) Geometry

local metric invariants (curvature bounds)	$\overset{?}{\longleftrightarrow}$	global topological invariants (homotopy type)

1. Riemannian case

M^n n-dim'l diff. manifold

$g_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ Riemannian metric

$d : M \times M \longrightarrow \mathbb{R}$ distance

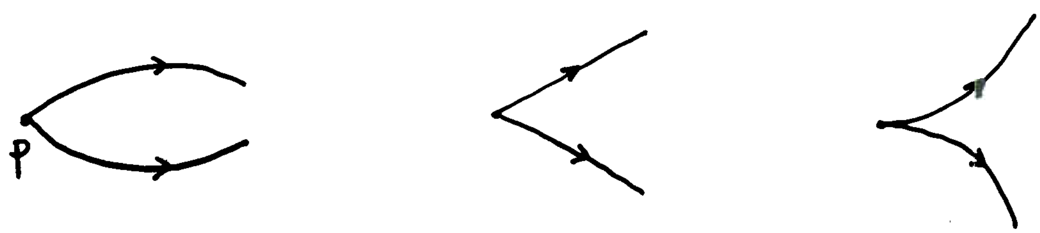
geodesics: curves which locally minimize distance

$K = K(\sigma) \in \mathbb{R}$ sectional curvature $\sigma^2 \subset T_xM$

controls the infinitesimal behaviour of geodesics.

Qualitative behaviour:

$K \geq 1$ $\qquad K \equiv 0$ $\qquad K \leq -1$



A simply-connected, complete Riemannian manifold with constant sectional curvature $K = 1$ is isometric to the standard sphere.

Question (H. Hopf): What happens, if $K \approx 1$?

Sphere Theorem (KLINGENBERG '61, BERGER, RAUCH, TOPONOGOV)
A simply-connected, complete Riemannian manifold with sectional curvature $\frac{1}{4} < K \leq 1$ is homeomorphic to the n-sphere.

The proof uses the injectivity radius estimate and the Toponogov triangle comparison theorem for the construction of the homeomorphism.

KLINGENBERG '63: A different proof using the injectivity radius estimate, Rauch comparison theorem and Morse theory for the energy functional on the space of free loops. (this proof carries over to the case of reversible Finsler metrics, DAZORD '68). This proof shows, that the manifold is homotopy equivalent to S^n.

2. Non-reversible Finsler metrics

Finsler metric $F : TM \longrightarrow \mathbb{R}^{\geq 0}$ s.t.

a) $F(\mu X) = \mu F(X), \quad \mu > 0$

b) $F(X) = 0 \Rightarrow X = 0$

c) $g^V(X,Y) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} F^2(V + sX + tY)$, g^V is positive definite

If $F(X) = \sqrt{g(X,X)}$, g Riemannian $\Rightarrow g^V = g$ for all V.

Definition: For a compact Finsler manifold the reversibility

$$\lambda = \lambda(M,F) := \max \left\{ \frac{F(-X)}{F(X)} \; ; \; X \in T_*M, X \neq 0 \right\}$$

$\lambda \geqq 1$ and $\lambda = 1 \Leftrightarrow F$ is reversible (symmetric), i.e. $F(X) = F(-X)$ for all X.

(non-symmetric) distance

$$\vartheta(p,q) := \min \{ L(c) \mid c \text{ curve joining } p \text{ and } q \}$$

Estimate

$$\frac{1}{\lambda} \vartheta(p,q) \leq \vartheta(q,p) \leqq \lambda \vartheta(p,q)$$

Symmetrized distance

$$d(p,q) := \frac{1}{2} \{ \vartheta(p,q) + \vartheta(q,p) \}$$

geodesics locally minimize ϑ

flag curvature $K(V;\sigma) \in \mathbb{R}$, (V,σ) flag, $0 \neq V \subset \sigma^2 \subset T_pM$ controls the behaviour of Jacobi fields

In contrast to the Riemannian case:

There are „many" (non-reversible) Finsler metrics of constant positive flag curvature:

FUNK '29, BRYANT '97 : projectively flat
BAO-SHEN '00, SHEN '01, ROBLES : Randers type, not projectively flat

Main results

$\ell = \ell(M,F)$: length of a shortest closed geodesic
$\mathrm{inj} = \mathrm{inj}(M,F)$: injectivity radius

Length estimate: Let (M,F) be a compact and simply-connected Finsler manifold with reversibility λ and $n = \dim(M) \geq 3$. If the flag curvature satisfies

$$\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$$

then

$$\ell \geq \pi\left(1 + \frac{1}{\lambda}\right), \quad \mathrm{inj} \geq \frac{\pi}{\lambda}$$

Examples: There are space forms with $K \equiv 1$ and reversibility λ, such that $\ell = \pi\left(1+\frac{1}{\lambda}\right)$ $\left(\mathrm{inj} \geq \frac{\pi}{\lambda}\right)$ (KATOK, SHEN)

SPHERE THEOREM A simply-connected, compact Finsler manifold of dimension $n \geq 3$ with reversibility λ, whose flag curvature satisfies

$$\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$$

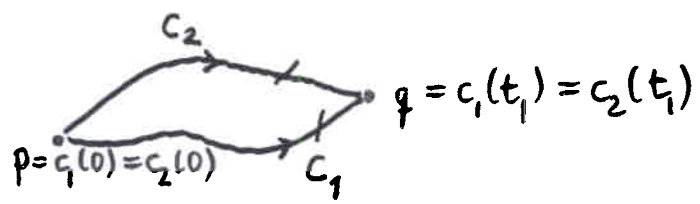
is homotopy equivalent to the n-sphere

$\lambda = 1$: reversible Finsler metric (DAZORD '68)
(Kern '71) differentiable sphere theorem for almost Riemannian Finsler metrics

Length estimate: Sketch of the proof

$c_1 : [0, \infty) \longrightarrow M$ is a geodesic, $F(c_1') = 1$, let
$t_1 := \sup \{ t > 0 \mid \vartheta(c(0), c(t)) = t \} < \infty$
then $c_1(t_1) \in Cut(p)$, i.e. $c_1(t_1)$ is a cut point and the cut locus $Cut(p)$ is the union of cut points of geodesics starting from $p = c_1(0)$.

If $t_1 < \pi$ then $c_1(t_1)$ is not conjugate to $\overbrace{p = c_1(0)}^{\text{(since } K \le 1)}$ along $c_1 | [0, t_1]$, hence there is a second minimal geodesic $c_2 : [0, t_1] \longrightarrow M$, $c_2(0) = p$, $c_2(t_1) = c_1(t_1)$.



Now assume $\ell < \pi(1 + \frac{1}{\lambda})$. We define two invariants:

$inj := \min \{ \vartheta(p,q) \mid q \in Cut(p),\ p \in M \}$ injectivity radius

$\hat{d} := \min \{ d(p,q) = \frac{1}{2}(\vartheta(p,q) + \vartheta(q,p)) \mid q \in Cut(p),\ p \in M \}$

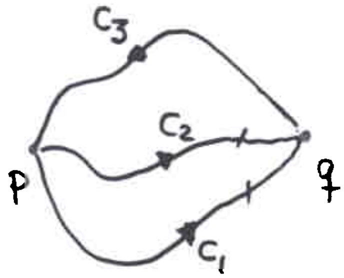
Hence

$$inj \geq \frac{2\hat{d}}{1+\lambda} \quad ; \quad \hat{d} \geq \frac{1}{2}\left(1 + \frac{1}{\lambda}\right) inj$$

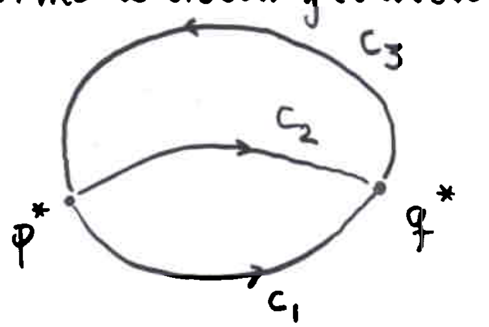
For $\lambda = 1$: $\quad inj = \hat{d}$

$\ell \le \pi(1 + \frac{1}{\lambda})$ implies: There are $p \in M$, $q \in Cut(p)$, $\vartheta(p,q) < \pi$

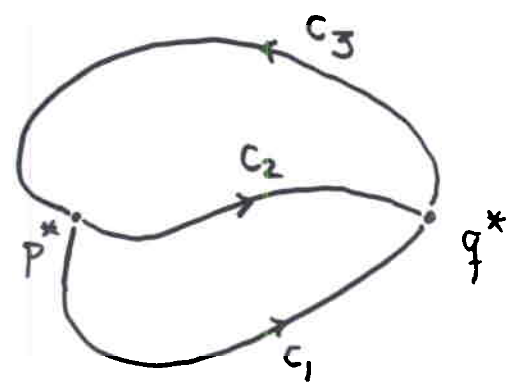
Geometrically: Let $p \in M$, $q \in Cut(p)$, $\vartheta(p,q) < \pi$.
Then there exist two minimal geodesic 2-gons $c_3 * c_1$, $c_3 * c_2$ with vertices p, q and length $2d(p,q)$.



The definition of $\hat{d} = \min\{d(p,q) \mid q \in Cut(p)\}$ implies:
If p^*, q^* satisfy: $\hat{d} = d(p^*, q^*)$, $q^* \in Cut(p)$ then one of the geodesic 2-gons is smooth at q^*, i.e. it forms a geodesic loop from p^* (but the proof does not show that it forms a closed geodesic)



or



Hence: There are $p^*, q^* \in Cut(p^*)$ and a geodesic loop γ with $\gamma(0) = p^*$, $\gamma(t_1) = q^*$, $t_1 = \vartheta(p^*, q^*)$ and
$L(\gamma) = 2\hat{d} = \ell$

Since $\pi_1 M = 0$ we can choose a homotopy

$$\gamma_s \in \Omega_p M := \{\sigma : [0,1] \to M \mid \sigma(0) = \sigma(1) = p, \sigma \text{ piecew. smooth}\}$$

between $\gamma = \gamma_0$ and the point curve $\gamma_1 = p$.
We can choose a homotopy of minimal energy

$$\eta := \max_{s \in [0,1]} E(\gamma_s).$$

We claim: $\eta < \frac{1}{2}\pi^2 (1 + \frac{1}{\lambda})^2$. This follows from the following 2 observations:

- The geodesic loops in $\Omega_p M$ are the critical points of the energy functional

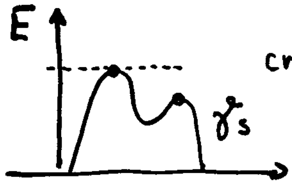
$$E : \Omega_p M \longrightarrow \mathbb{R} \; ; \; E(\sigma) = \frac{1}{2} \int_0^1 F^2(\sigma'(t))\, dt$$

- From Rauch comparison one concludes, that a geodesic loop σ with $L(\sigma) \geq \pi(1 + \frac{1}{\lambda})$ has a Morse-index $\mathrm{ind}_\Omega(\sigma) \geq n - 1 \geq 2$ since

$$K \geq \left(1 - \frac{1}{1+\lambda}\right)^2$$

(since a great circle on a sphere of constant curvature $\left(1 - \frac{1}{1+\lambda}\right)^2$ has length $\frac{\pi}{1 - \frac{1}{1+\lambda}} = \pi(1 + \frac{1}{\lambda})$)

picture in $\Omega_p M$:



critical values (one can „push" γ_s below the critical value of geodesic loops with $\mathrm{ind}_\Omega \geq 2$)

Since $\eta < \pi(1+\frac{1}{\lambda})$ we can lift the homotopy γ_s via the exponential map

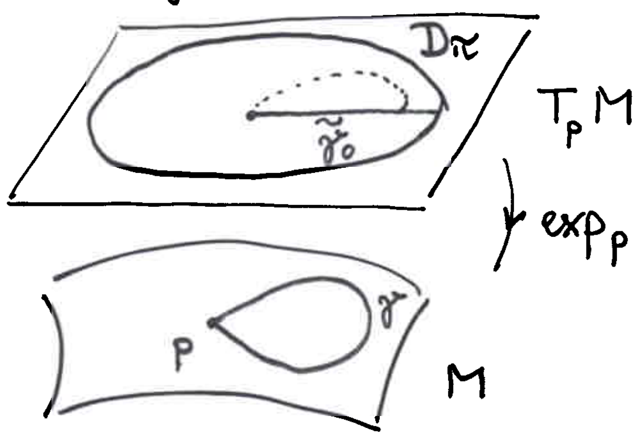
$$\exp_p : D_\pi := \{ X \in T_pM \mid F(X) \leq \pi \} \longrightarrow M$$

which has maximal rank, since $K \leq 1$. Hence there is a homotopy

$$\tilde{\gamma}_s : [0,1] \longrightarrow D_\pi \; ; \; \tilde{\gamma}_s(0) = \tilde{\gamma}_s(1) = 0$$

with
$$\gamma_s = \exp_p \circ \tilde{\gamma}_s \, .$$

But since $\gamma(t) = \gamma_0(t) = \exp_p(t\gamma'(0))$ for all $t \leq \pi / F(\gamma'(0))$ this is a contradiction.



<u>Proof of the sphere theorem</u> (sketch)

$c : S^1 \longrightarrow M$ closed geodesic, $L(c) \geq \pi(1+\frac{1}{\lambda})$

$\Rightarrow$ Morse index $\mathrm{ind}_\Omega(c) \geq n-1$

Then the Morse inequalities show that the free loop space is $(n-2)$-connected and hence the manifold is $(n-1)$-connected.

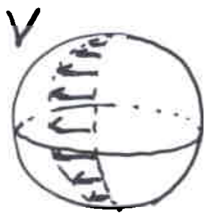
Example:

(KATOK'73, ZILLER'82)

standard 2-sphere (S^2, g)

Killing field $V(x,y,z) = (-y, x, 0) = \frac{d}{dt}\Big|_{t=0} R(t)(x,y,z)$

$$R(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Rotation around north pole with angle t



$H_0 : T^*S^2 \longrightarrow \mathbb{R}$, $H_0(y) = \sqrt{g^*(y,y)}$

with hamiltonian flow φ^t

$G : T^*S^2 \longrightarrow \mathbb{R}$, $G(y) = y(V)$

with hamiltonian flow $\zeta^t = R(t)^*$.

Define a quadratic Hamiltonian $\mathcal{H}_\varepsilon := \frac{1}{2} H_\varepsilon^2$

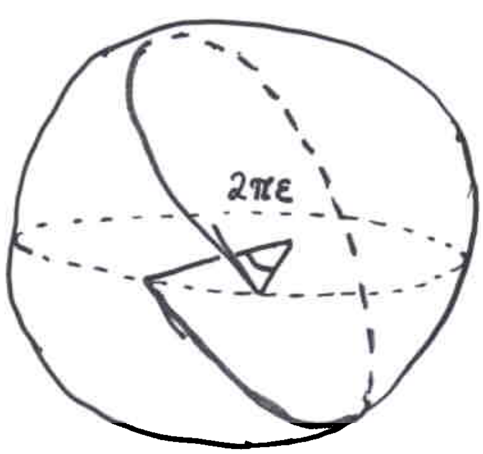
$$H_\varepsilon(y) := H_0(y) + \varepsilon\, G(y), \quad \varepsilon \in [0,1)$$

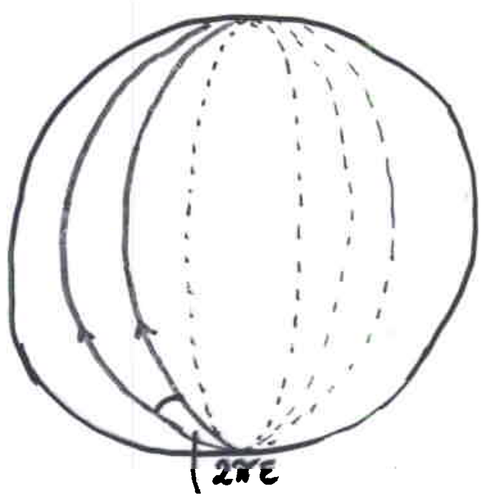
with Legendre-transformation $L_\varepsilon : T^*S^2 \longrightarrow TS^2$ and corresponding Finsler metric

$$F_\varepsilon := H_\varepsilon \circ L_\varepsilon^{-1}$$

The flows φ^t, ζ^t commute, since $R(t)$ are isometries, hence the hamiltonian flow of H_ε:

$$\psi^t = \varphi^t \circ \zeta^t$$





If $\varepsilon \in \mathbb{R} - \mathbb{Q}$ then there are only two geometrically distinct closed geodesics $c_\pm(t) = (\cos t, \pm \sin t, 0)$

A Hamiltonian function of Randers type corresponds to a Finsler metric of Randers type (HRIMIUC, SHIMADA '96), in geodesic polar coordinates $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$ of the standard metric:

$$H_\varepsilon(r, \theta, p_1, p_2) = \sqrt{p_1^2 + \frac{p_2^2}{\sin^2 r}} + \varepsilon p_2$$

$$F_\varepsilon(r, \theta, \dot{q}_1, \dot{q}_2) = \frac{\sqrt{(1 - \varepsilon^2 \sin^2 r)\, \dot{q}_1^2 + \sin^2(r)\, \dot{q}_2^2} - \varepsilon \sin^2 r\, \dot{q}_2^2}{1 - \varepsilon^2 \sin^2 r}$$

It follows from (SHEN '01) that F_ε has constant flag curvature 1.

Since $L(c_\pm) = \dfrac{2\pi}{1 \pm \varepsilon}$, $\lambda = \dfrac{1+\varepsilon}{1-\varepsilon}$ the shortest closed geodesic has length $L(c_+) = \pi\left(1 + \frac{1}{\lambda}\right)$.

One can derive estimates for the number of (geometrically distinct) closed geodesics.

Definition. For a closed geodesic c define the average index $\alpha_c = \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m} = \lim_{m\to\infty} \frac{\#\,\text{conj. pts. along } c^m}{m}$

and the mean average index. $\overline{\alpha}_c = \frac{\alpha_c}{L(c)}$

Theorem (R. '94). If M has the rational homotopy type of a sphere or a projective space (over $\mathbb{C}, \mathbb{H}$)
We assume that the Finsler metric (M, F) has only finitely many closed geodesic $(c_1, c_2, \dots, c_N, \dots\ )$
Then there is $\sigma \in \mathbb{R}$ and

$$\sum_{\substack{\{c \text{ closed geodesic} \\ \overline{\alpha}_c = \sigma\ \}}} \frac{1}{\alpha_c} \geq \frac{1}{2}$$

As a consequence:
Corrollary: a) For every Finsler metric on S^{2n} with $\frac{1}{4} < K \leq 1$ there are at least $\frac{n-1}{4}$ geometrically district closed geodesics (which are resonant)
b) For every Finsler metric on S^{2n} or $\mathbb{C}P^{2n}$ with $\left(\frac{2}{n-1}\right)^2 \leq K \leq 1$
there are at least two geometrically distinct closed geodesics.