THE GENERALITY OF FINSLER METRICS WITH CONSTANT FLAG CURVATURE AND SOME EXOTIC HOLONOMY GROUPS

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Background: Let M^{n+1} be a connected, smooth manifold and let $\Sigma^{2n+1} \subset TM$ be a (generalized) Finsler structure on M:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota} & TM \\ & & \downarrow \pi \\ & & M \end{array} \qquad \Rightarrow \qquad \begin{array}{ccc} \mathrm{O}(n) & \xrightarrow{\iota} & F \\ & & \downarrow \pi \\ & & \Sigma \end{array}$$

First structure equations:

$$\begin{aligned} \mathrm{d}\omega_0 &= \qquad\qquad -\theta_{0j} \wedge \omega_j\,, \qquad \text{(the Hilbert form)} \\ \mathrm{d}\omega_i &= \theta_{0i} \wedge \omega_0 - \theta_{ij} \wedge \omega_j - I_{ijk}\,\theta_{0k} \wedge \omega_j\,, \\ \mathrm{d}\theta_{0i} &= -\theta_{ij} \wedge \theta_{0j} + R_{0i0k}\,\omega_0 \wedge \omega_k + \tfrac{1}{2} R_{0ijk}\,\omega_j \wedge \omega_k + J_{ijk}\,\theta_{0k} \wedge \omega_j \end{aligned}$$

Constant Flag Curvature c: $R_{0i0j} = c\,\delta_{ij}$. ($\Rightarrow R_{0ijk} \equiv 0$)

Question: How 'general' are these (gen.) Finsler structures?

I will concentrate on the case $c = 1$ in this lecture.

Simplified structure equations:

$$\begin{aligned} \mathrm{d}\omega_0 &= \qquad\qquad -\theta_{0j}\wedge\omega_j \\ \mathrm{d}\omega_i &= \theta_{0i}\wedge\omega_0 - \theta_{ij}\wedge\omega_j - I_{ijk}\,\theta_{0k}\wedge\omega_j\,, \\ \mathrm{d}\theta_{0i} &= -\omega_i\wedge\omega_0 - \theta_{ij}\wedge\theta_{0j} + J_{ijk}\,\theta_{0k}\wedge\omega_j \end{aligned}$$

Prop: (B—, Bejancu & Farran) Let E be the Reeb vector field on Σ (i.e., the vector field dual to ω_0, the Hilbert form), then

$$\omega_1{}^2 + \cdots + \omega_n{}^2 + \theta_{01}{}^2 + \cdots + \theta_{0n}{}^2$$

is invariant under the flow of E.

Defn: Let Q be the space of integral curves of E. Say that Σ is geodesically simple if Q has a Haus. manifold str. so that the projection $\ell : \Sigma \to Q$ is a smooth submersion.

Theorem: (B—) The space Q is naturally a Kähler manifold with Kähler metric and 2-form satisfying

$$\ell^*\bigl(\mathrm{d}\sigma^2\bigr) = {\omega_1}^2 + \cdots + {\omega_n}^2 + {\theta_{01}}^2 + \cdots + {\theta_{0n}}^2 \,,$$

$$\ell^*\bigl(\Omega\bigr) = \,-\,\omega_j \wedge \theta_{0j} = -\mathrm{d}\omega_0 \,.$$

PROOF: Write $\zeta_i = \omega_i - \mathrm{i}\,\theta_{0i}$, so that

$$\ell^*(\mathrm{d}\sigma^2) = \zeta_1 \circ \overline{\zeta_1} + \cdots + \zeta_n \circ \overline{\zeta_n}\,, \quad \text{and} \quad \ell^*(\Omega) = \tfrac{\mathrm{i}}{2}\,\zeta_i \wedge \overline{\zeta_i}\,.$$

The structure equations imply

$$\mathrm{d}\zeta_i = -\mathrm{i}\,\omega_0 \wedge \zeta_i - \theta_{ij} \wedge \zeta_j + \tfrac{\mathrm{i}}{2}(I_{ijk} + \mathrm{i}\,J_{ijk})\,\overline{\zeta_j} \wedge \zeta_k\,,$$

so the Newlander-Nirenberg theorem implies that there is an integrable complex structure on Q for which $\{\zeta_1, \ldots, \zeta_n\}$ spans the ℓ-pullbacks of the $(1,0)$-forms.

Since $\ell^*(\Omega) = -\mathrm{d}\omega_0$ is closed and ℓ is a submersion, Ω is also closed. Thus, $\bigl(\mathrm{d}\sigma^2, \Omega\bigr)$ defines a Kähler structure on Q. □

REMARK: A Kähler structure is just a torsion-free $\mathrm{U}(n)$-structure.

A finer structure. Consider $\zeta = \big(\omega_i - \mathrm{i}\,\theta_{0i}\big) = (\zeta_i) : TF \to \mathbb{C}^n$.

$\zeta(v) = 0$ iff $q'(v) = 0$ for $q : F \to \Sigma \to Q$ (the composition).

Define $v(f) : T_{q(f)}Q \to \mathbb{C}^n$ so that this diagram commutes:

$$\begin{array}{ccc} T_f F & \xrightarrow{\ \zeta_f\ } & \mathbb{C}^n \\ \big\downarrow & \nearrow & \\ T_{q(f)}Q & & \end{array}$$

Prop: v maps F into an open subset of an $S^1\!\cdot \mathrm{O}(n)$-structure on Q, where

$$S^1\!\cdot \mathrm{O}(n) = \left\{ \ \mathrm{e}^{\mathrm{i}\alpha} A \ \middle| \ \mathrm{e}^{\mathrm{i}\alpha} \in S^1,\ A \in \mathrm{O}(n) \ \right\} \subset \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C}).$$

Prop: The $S^1\!\cdot \mathrm{O}(n)$-structure is torsion-free iff Σ is Riemannian, but the underlying $S^1\!\cdot \mathrm{GL}(n, \mathbb{R})$-structure is always torsion-free.

REMARK: For $n > 1$, the group $S^1\!\cdot \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$ is not on the accepted list of groups that can be holonomy of an irreducible torsion-free connection in dimension $2n$!

The surface case: A double fibration:

$$\begin{array}{ccc} & \Sigma^3 & \\ {}^{\ell}\swarrow & & \searrow{}^{\pi} \\ Q^2 & & M^2 \end{array}$$

$$\ell^*(\mathrm{d}\sigma^2) = {\omega_1}^2 + {\theta_{01}}^2\,, \qquad \ell^*(\Omega) = \theta_{01}\wedge\omega_1\,.$$

Extra structure: The 1-form β:

$$\exists\beta\in\Omega^1(Q) \text{ so that } \ell^*\beta = -I_{111}\,\omega_1 + J_{111}\,\theta_{01}.$$

Prop: Let K be the Gauss curvature of $\mathrm{d}\sigma^2$. Then $\mathrm{d}\beta = (1-K)\,\Omega$ and, for $x\in M$, the curve $C_x = \ell\bigl(\pi^{-1}(x)\bigr)$ is a β-geodesic.

Defn: If $(Q, \mathrm{d}\sigma^2)$ is an oriented surface, with area form Ω and β is a 1-form on Q, a curve $C\subset Q$ is a β-geodesic if $\kappa_C\,ds_C = C^*\beta$.

$$
\begin{array}{ccc}
 & \Sigma^3 & \\
 {}^{\ell}\swarrow & & \searrow^{\pi} \\
Q^2 & & M^2
\end{array}
$$

Prop: (Converse) If $(Q, \mathrm{d}\sigma^2, \Omega, \beta)$ satisfies $\mathrm{d}\beta = (1-K)\,\Omega$ and if $\ell : \Sigma \to Q$ is the $\mathrm{d}\sigma^2$-unit sphere bundle, then Σ is foliated by β-geodesics and the leaf space M carries a canonical (generalized) Finsler structure of constant flag curvature $+1$.

Cor: (Local generality) The Finsler surfaces of constant flag curvature $+1$ depend on two arbitrary functions of two variables, up to diffeomorphism.

Prop: (Global) If $(S^2, \mathrm{d}\sigma^2, \Omega, \beta)$ satisfies $\mathrm{d}\beta = (1-K)\,\Omega$ and if all of the β-geodesics are closed, then it comes from a global Finsler structure with constant flag curvature $+1$ on $M = S^2$.

Lemma: Let $(Q, \mathrm{d}\sigma^2, \Omega, \beta)$ be an oriented surface with 1-form and L a positive function on Q. Set

$$\mathrm{d}\tilde{\sigma}^2 = L\,\mathrm{d}\sigma^2, \qquad \tilde{\Omega} = L\,\Omega, \qquad \tilde{\beta} = \beta + *\mathrm{d}(\log\sqrt{L}).$$

Then the $\tilde{\beta}$-geodesics with respect to $\left(\mathrm{d}\tilde{\sigma}^2, \tilde{\Omega}\right)$ are the same as the β-geodesics of $\left(\mathrm{d}\sigma^2, \Omega\right)$.

Lemma: Let Q be a surface endowed with a metric $\mathrm{d}\sigma^2$ with Gauss curvature $K > 0$ and area form Ω. Then the data

$$\mathrm{d}\bar{\sigma}^2 = K\,\mathrm{d}\sigma^2\,, \qquad \bar{\Omega} = K\,\Omega\,, \qquad \bar{\beta} = *\mathrm{d}\big(\log\sqrt{K}\big)\,,$$

satisfy $\mathrm{d}\bar{\beta} = (1 - \bar{K})\,\bar{\Omega}$, where $\bar{K}$ is the Gauss curvature of $\mathrm{d}\bar{\sigma}^2$.

Theorem: If $\mathrm{d}\sigma_0^2$ is a Zoll metric on $Q = S^2$ with area form Ω_0 and positive Gauss curvature K_0. Let $M \simeq S^2$ be the space of oriented $\mathrm{d}\sigma_0^2$-geodesics on Q. Then the data

$$\mathrm{d}\sigma^2 = K_0\,\mathrm{d}\sigma_0^2\,, \qquad \Omega = K_0\,\Omega\,, \qquad \beta = *\mathrm{d}\Big(\log\sqrt{K_0}\Big)$$

come from a Finsler metric on M with constant flag curvature $+1$.

Higher dimensions. From now on, assume $n > 1$.

Recall the structure equations of the $\mathrm{O}(n)$-structure $u : F \to \Sigma$:

$$\begin{aligned} \mathrm{d}\omega_0 &= \phantom{\theta_{0i}\wedge\omega_0} -\theta_{0j}\wedge\omega_j \\ \mathrm{d}\omega_i &= \theta_{0i}\wedge\omega_0 - \theta_{ij}\wedge\omega_j - I_{ijk}\,\theta_{0k}\wedge\omega_j\,, \\ \mathrm{d}\theta_{0i} &= -\omega_i\wedge\omega_0 - \theta_{ij}\wedge\theta_{0j} + J_{ijk}\,\theta_{0k}\wedge\omega_j \end{aligned}$$

and how $\zeta = (\zeta_i) = \big(\omega_i - \mathrm{i}\,\theta_{0i}\big)$ defines a $S^1\!\cdot\mathrm{O}(n)$-structure on Q: An $f \in F$ defines an isomorphism $\upsilon(f) : T_{q(f)}Q \to \mathbb{C}^n$. Although

$$\mathrm{d}\zeta_i = -\mathrm{i}\,\omega_0\wedge\zeta_i - \theta_{ij}\wedge\zeta_j + \tfrac{\mathrm{i}}{2}(I_{ijk} + \mathrm{i}\,J_{ijk})\,\overline{\zeta_j}\wedge\zeta_k\,,$$

shows that this $S^1\!\cdot\mathrm{O}(n)$-structure has torsion, writing

$$\sigma_{ij} = \sigma_{ji} = \overline{\sigma_{ij}} = \tfrac{\mathrm{i}}{2}(I_{ijk} - \mathrm{i}\,J_{ijk})\,\zeta_j - \tfrac{\mathrm{i}}{2}(I_{ijk} + \mathrm{i}\,J_{ijk})\,\overline{\zeta_j}$$

shows that

$$\mathrm{d}\zeta_i = -\big(\mathrm{i}\,\omega_0 + \theta_{ij} + \sigma_{ij}\big)\wedge\zeta_j = -\big(\mathrm{i}\,\omega_0 + \phi_{ij}\big)\wedge\zeta_j\,,$$

so the underlying $S^1\!\cdot\mathrm{GL}(n,\mathbb{R})$-structure on Q is torsion-free.

Now $R(f) = v(f)^{-1}(\mathbb{R}^n) \subset T_{q(f)}Q$ depends only on $u(f) \in \Sigma$, so the $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure on Q defines an S^1-bundle of n-planes $R \subset \mathrm{Gr}(n, TQ)$:

$$\begin{array}{ccccc} R^{2n+1} & & \longleftarrow & & \Sigma^{2n+1} \\ & \searrow & & \ell \swarrow & & \searrow^{\pi} \\ & & Q^{2n} & & & & M^{n+1} \end{array}$$

Prop: The images $C_x = \ell\big(\pi^{-1}(x)\big) \subset Q$ have the n-planes in R as their tangent spaces. Conversely, a connected $C^n \subset Q$ whose tangent planes belong to R lies in a unique C_x.

Defn: A torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure on a $2n$-manifold Q will be said to be R-integrable if every n-plane $E \in R$ is tangent to an n-manifold $C \subset Q$ whose tangent spaces belong to R.

Prop: When $n > 2$, any torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure on a $2n$-manifold Q is R-integrable (and hence M^{n+1} exists).

The structure equations. Now let $q : F \to Q$ be a torsion-free, R-integrable $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-structure. The first structure equation

$$\mathrm{d}\zeta^i = -\big(\mathrm{i}\,\delta^i_j\,\omega_0 + \phi^i_j\big) \wedge \zeta^j$$

implies there are $\mathbb{R}$-valued functions $b_{ij} = b_{ji}$ and $r^i_{jkl} = r^i_{kjl} = r^i_{jlk}$ on F satisfying the second structure equation:

$$\mathrm{d}\omega_0 = -\mathrm{i}\,b_{kl}\,\zeta^k \wedge \overline{\zeta^l}\,,$$

$$\mathrm{d}\phi^i_j + \phi^i_k \wedge \phi^k_j = b_{jl}\left(\zeta^i \wedge \overline{\zeta^l} + \overline{\zeta^i} \wedge \zeta^l\,\right) + \mathrm{i}\,r^i_{jkl}\,\zeta^k \wedge \overline{\zeta^l}\,.$$

We will also need the second Bianchi identity for such structures: There exist unique $\mathbb{C}$-valued functions $B_{ijk} = B_{jik} = B_{ikj}$ and $R^i_{jklm} = R^i_{kjlm} = R^i_{jlkm} = R^i_{jkml}$ on F so that

$$\begin{aligned}
\mathrm{d}b_{ij} &= b_{kj}\phi^k_i + b_{ik}\phi^k_j + \mathrm{Re}\left(B_{ijk}\zeta^k\right)\,,\\
\mathrm{d}r^i_{jkl} &= -r^m_{jkl}\phi^i_m + r^i_{mkl}\phi^m_j + r^i_{jml}\phi^m_k + r^i_{jkm}\phi^m_l\\
&\qquad + \mathrm{Re}\Big(\big(R^i_{jklm} - \mathrm{i}\,(\delta^i_j\,B_{klm} + \delta^i_k\,B_{ljm} + \delta^i_l\,B_{kjm})\big)\,\zeta^m\Big)\,.
\end{aligned}$$

Prop: (B—) The 2^{nd} Bianchi tableau for torsion-free, R-integrable $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-structures is involutive, with Cartan characters given by

$$s_k = \begin{cases} 0, & k = 0, 1, \\ k - 1 + n\big(n + (n{+}1{-}k)(k{-}2)\big), & 2 \le k \le n{+}1, \\ 0, & n{+}1 < k \le 2n. \end{cases}$$

Theorem: (B—) Up to diffeomorphism, the local torsion-free, R-integrable $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-structures depend on $n(n{+}1)$ functions of $n{+}1$ variables. The curvature can be freely specified at a point.

Cor: (B—) The subgroup $S^1 \cdot \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R})$ *does* occur as the holonomy of a torsion-free affine connection in dimension $2n$ (even though it was omitted from the classification list given by Schwachhöfer and Merkulov).

Recovering the Finsler structure. Let $q : F \to Q^{2n}$ be a torsion-free, R-integrable $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure, with str. eqs.

$$\begin{aligned} \mathrm{d}\zeta^i &= -\big(\mathrm{i}\,\delta^i_j\,\omega_0 + \phi^i_j\big) \wedge \zeta^j \\ \mathrm{d}\omega_0 &= -\mathrm{i}\,b_{kl}\,\zeta^k \wedge \overline{\zeta^l}\,, \\ \mathrm{d}\phi^i_j + \phi^i_k \wedge \phi^k_j &= b_{jl}\left(\zeta^i \wedge \overline{\zeta^l} + \overline{\zeta^i} \wedge \zeta^l\,\right) + \mathrm{i}\,r^i_{jkl}\,\zeta^k \wedge \overline{\zeta^l}\,. \end{aligned}$$

If the real symmetric matrix $b = (b_{ij})$ is positive definite, then the equation $b_{ij} = \frac{1}{2}\delta_{ij}$ defines an $S^1 \cdot \mathrm{O}(n)$-structure $F_0 \subset F$ and the structure equations show that it comes from a generalized Finsler structure with constant flag curvature $+1$ on the space M^{n+1} of R-leaves of the structure F.

$$\begin{array}{ccccc} F & \longrightarrow & R^{2n+1} & & \\ & \searrow & {}^{\ell}\swarrow & \searrow^{\pi} & \\ & Q^{2n} & & & M^{n+1} \end{array}$$