Ricci Curvature in Finsler Geometry

David Bao
Joint work with
Colleen Robles

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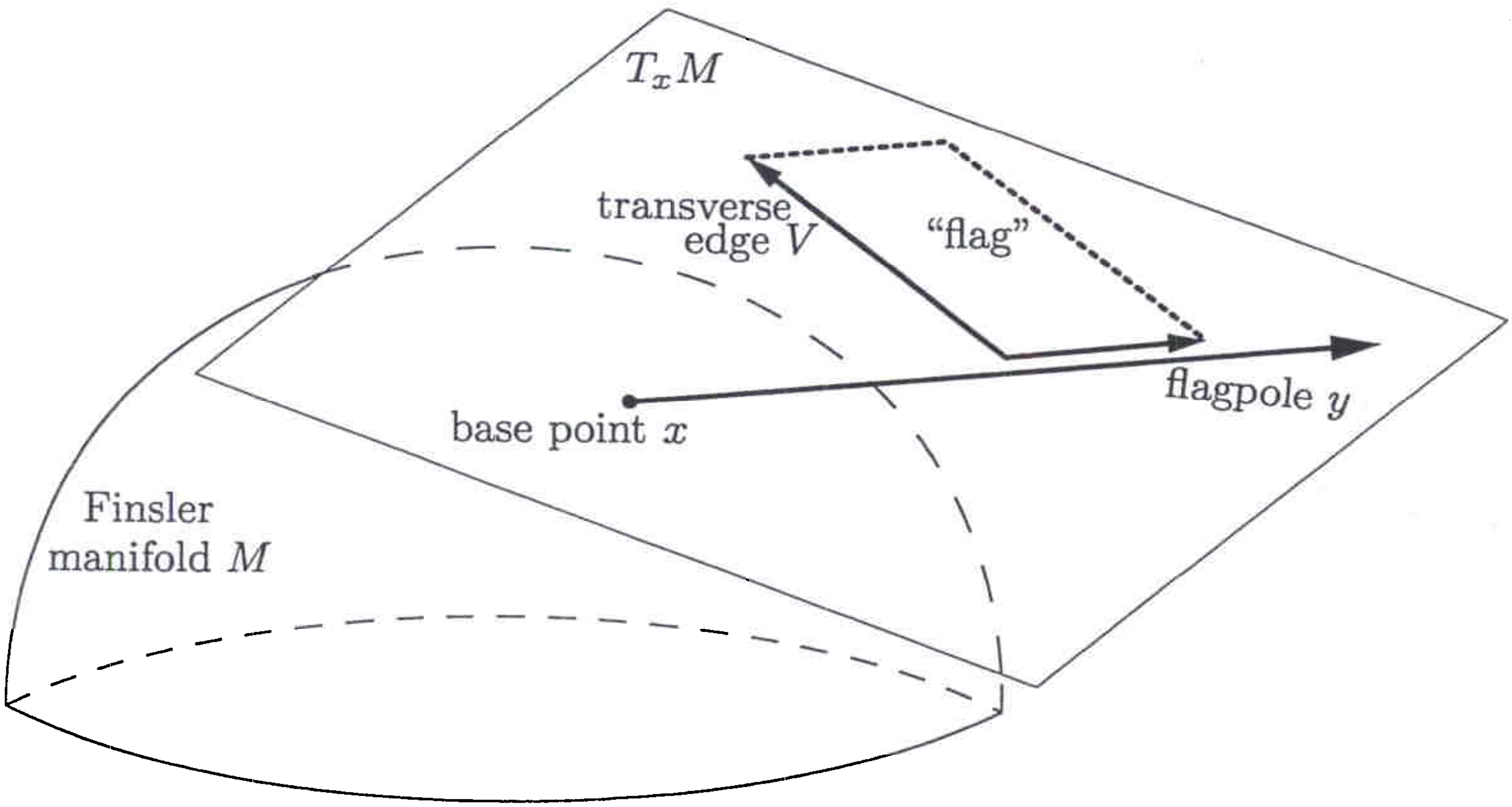

T_xM
transverse
edge V
"flag"
flagpole y
base point x
Finsler
manifold M

Bao 2

Sphere's worth of inner products

$$g_y := \left[\tfrac{1}{2}F^2(x,y)\right]_{y^i y^j} dx^i \otimes dx^j$$

Unit transverse edges orthogonal to the flagpole

$$E_\alpha \qquad \alpha = 1, \ldots, n{-}1$$

set $E_n = \frac{y}{F}$

Flag Curvature $K(x, y, V)$

$$\frac{V^i \left(y^j R_{jikl} \, y^l \right) V^k}{\{F(y)\}^2 \, g_y(V, V) - \{g_y(y, V)\}^2}$$

with $V = E_\alpha$

$$(E_\alpha)^i \left\{ (E_n)^j R_{jikl} \, (E_n)^l \right\} (E_\alpha)^k$$

$$\|$$

$$R_{n\alpha\alpha n} \equiv R_{\alpha\alpha}$$

Ricci scalar

$$Ric := \sum_\alpha K(x, y, E_\alpha) = R^i{}_i$$

Ricci tensor Ric_{ij}

$$\left(\tfrac{1}{2} F^2 Ric\right)_{y^i y^j} = \left(\tfrac{1}{2}\, y^p\, R_p{}^k{}_{kq}\, y^q\right)_{y^i y^j}$$

- F **Riem.** $\Rightarrow \quad Ric_{ij} = R_i{}^k{}_{kj}$

Bonnet-Myers Theorem

Hypotheses:

- dim $M = n$
- F forward geodesically complete
- $Ric \geqslant (n-1)\lambda > 0$

Conclusions:

- Geods. of length $\geqslant \frac{\pi}{\sqrt{\lambda}}$ have conj. pts.
- diam$(M) \leqslant \frac{\pi}{\sqrt{\lambda}}$
- M is compact
- $\pi(M, x)$ is finite

Einstein metrics

$$Ric = \lambda(x)$$

$$\Updownarrow$$

$$Ric_{ij} = \lambda(x)\, g_{ij}$$

- $Ric = Ric_{ij}\ \frac{y^i}{F} \frac{y^j}{F}$

- $Ric_{ij} = Ric\, g_{ij} + O(\, [Ric]_{y,yy}\,)$

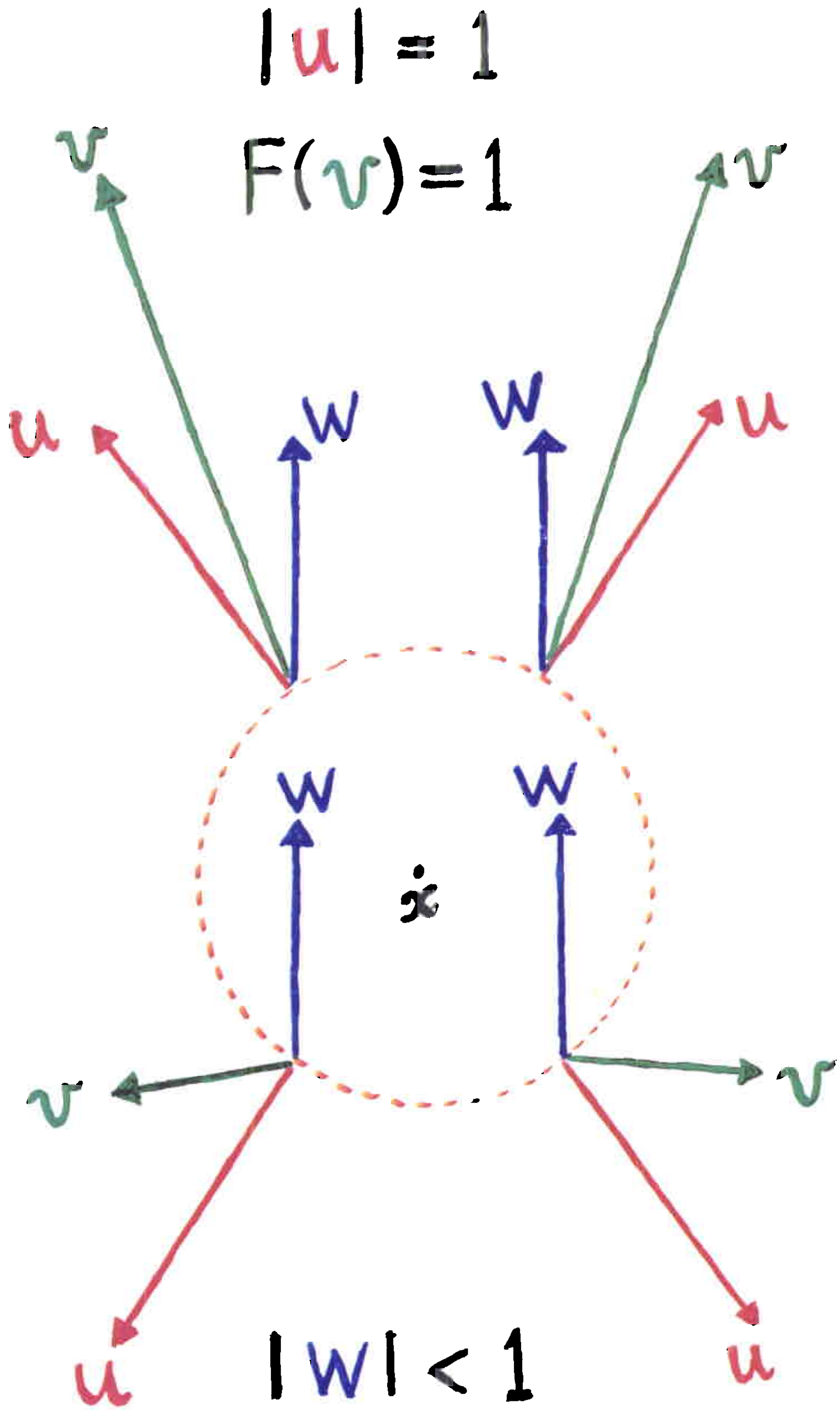
Randers metrics $F = \alpha + \beta$

- $\alpha := \sqrt{\tilde{a}_{ij(x)} y^i y^j}$
- $\beta := \tilde{b}_{i(x)} y^i$

F and g_y positive definite

$$\Updownarrow$$

$$\| \tilde{b} \| < 1$$



Bao 9

- **Start with** $|u| = 1$:

$$|v|^2 - 2\, v \bullet W + |W|^2 = 1$$

- **Express** $|v|$ **as** $P + Q$:

$$|W| \cos\theta \;+\; \sqrt{\;|W|^2 \cos^2\theta + \left\{1 - |W|^2\right\}}$$

- **Use the fact** $F(v) = 1$:

$$F(v) = 1 = |v| \frac{1}{Q + P} = |v| \frac{Q - P}{Q^2 - P^2}$$

- **Homogeneity** $F(y) = F(\lambda v) = \lambda\, F(v)$:

$$\frac{\sqrt{(y \bullet W)^2 + |y|^2 \left\{1 - |W|^2\right\}}}{1 - |W|^2} + \frac{-y \bullet W}{1 - |W|^2}$$

The Shen Perturbation

New **Riem. metric** $\tilde{a}$**:**

$$\frac{(y \bullet z)\left\{1 - |W|^2\right\} + (y \bullet W)(z \bullet W)}{\{\ 1 - |W|^2\ \}^2}$$

Drift 1-form $\tilde{b}$**:**

$$\frac{-y \bullet W}{1 - |W|^2}$$

Strong Convexity:

$$\|\,\tilde{b}\,\| \ = \ |W| \ < \ 1$$

Key *Tensors*

- $\textbf{lie}_{ij} := \tilde{b}_{i|j} + \tilde{b}_{j|i}$

$$\boxed{\textbf{lie} \;=\; \mathcal{L}_{\tilde{b}}\,\tilde{a}}$$

- $\textbf{curl}_{ij} := \tilde{b}_{i|j} - \tilde{b}_{j|i}$

$$\boxed{\textbf{curl} \;=\; -\,d\,\tilde{b}}$$

- $\theta_j := \tilde{b}^i \textbf{curl}_{ij}$

$$\boxed{\theta \;=\; -\,i_{\tilde{b}}\left\{d\,\tilde{b}\right\}}$$

Randers metric has $Ric = (n-1)K(x)$

$\Updownarrow$

$\exists$ a constant σ

such that the following PDEs hold:

- **Basic Equation**

$$\mathcal{L}_{\tilde{b}}\, \tilde{a} = \sigma \left(\tilde{a} - \tilde{b} \otimes \tilde{b}\right) - \left(\tilde{b} \otimes \theta + \theta \otimes \tilde{b}\right)$$

- **E(23) Equation**

$$\frac{8}{n-1} \tilde{\nabla} \cdot \mathbf{curl} = \left(\sigma^2 + 16K\right) \tilde{b} + 4\,\sigma\theta + 4\, i_\theta \mathbf{curl}$$

- **Curvature Equation for $\widetilde{Ric}_{ij}$**

$$\begin{aligned}
&+ \left\{ (n-1)\left(K - \tfrac{3}{16}\sigma^2\right) - \tfrac{1}{4}\,\|\mathbf{curl}\|^2 \right\} \tilde{a}_{ij} \\
&+ (n-1)\left(K + \tfrac{1}{16}\sigma^2\right) \tilde{b}_i\, \tilde{b}_j - \tfrac{1}{2}\, \mathbf{curl}^s{}_i\, \mathbf{curl}_{sj} \\
&- (n-1)\,\tfrac{1}{4} \left\{\theta_i\, \theta_j + \theta_{i|j} + \theta_{j|i}\right\}
\end{aligned}$$

$$\sigma = \frac{2\,\tilde{\nabla} \cdot \tilde{b}}{n - \|\tilde{b}\|^2}$$

Example in dim 4

- Shen perturb the standard Riemannian metric on the product $S^2 \times S^2$

- Use $W := (\partial_\theta, 0)$, where ∂_θ is tangent to the latitudes on the first factor S^2

- The resulting Randers metric has $Ric = 1$, but is *not* of constant flag curvature.

$$\alpha := \mathrm{sqrt}((-2\,u\,y^3\,x\,v - 2\,u\,y\,v\,x^3 + 2\,u^2\,y^2\,z^4 + 2\,u^2\,y^2\,t^4 + 4\,v^2\,x^2\,t^2 + 2\,v^2\,x^2\,z^4 + 2\,v^2\,x^2\,t^4$$
$$+ 2\,w^2\,x^2\,t^2 + 2\,w^2\,y^2\,t^2 + w^2\,t^2\,x^4 + w^2\,t^2\,y^4 + 2\,s^2\,x^2\,y^2 + 2\,s^2\,x^2\,z^2 + 2\,s^2\,y^2\,z^2 + s^2\,z^2\,x^4$$
$$+ s^2\,z^2\,y^4 + 2\,u^2\,z^2 + 2\,u^2\,t^2 + u^2\,z^4 + u^2\,t^4 + 2\,v^2\,z^2 + 2\,v^2\,t^2 + v^2\,z^4 + v^2\,t^4 + 2\,w^2\,x^2$$
$$+ 2\,w^2\,y^2 + w^2\,x^4 + w^2\,y^4 + 4\,u^2\,y^2\,t^2 - 4\,x\,v\,y\,u - 2\,z\,s\,t\,w + 2\,x^2\,v^2 + u^2 + v^2 + 2\,y^2\,u^2$$
$$+ z^2\,s^2 + t^2\,w^2 + 2\,v^2\,z^2\,t^2 + w^2 + s^2 + 4\,u^2\,y^2\,z^2 + 2\,w^2\,x^2\,y^2 + 4\,v^2\,x^2\,z^2 + 2\,u^2\,z^2\,t^2$$
$$+ 4\,u^2\,y^2\,z^2\,t^2 + 4\,v^2\,x^2\,z^2\,t^2 - 8\,x\,v\,y\,u\,z^2 - 8\,x\,v\,y\,u\,t^2 - 4\,x\,v\,y\,u\,z^4 - 8\,x\,v\,y\,u\,z^2\,t^2$$
$$- 4\,x\,v\,y\,u\,t^4 + 2\,x^2\,y^2\,v^2\,z^2\,t^2 + 2\,s^2\,x^2 + 2\,s^2\,y^2 + s^2\,x^4 + s^2\,y^4 + 2\,w^2\,t^2\,x^2\,y^2 + 2\,s^2\,z^2\,x^2\,y^2$$
$$+ 2\,u^2\,y^2\,x^2\,z^2\,t^2 + 2\,x^2\,y^2\,v^2\,z^2 + 2\,x^2\,y^2\,v^2\,t^2 + x^2\,y^2\,v^2\,z^4 + x^2\,y^2\,v^2\,t^4 + 2\,u^2\,y^4\,z^2\,t^2$$
$$+ 2\,v^2\,x^4\,z^2\,t^2 - 4\,z\,s\,t\,w\,x^2 - 4\,z\,s\,t\,w\,y^2 - 2\,z\,s\,t\,w\,x^4 - 4\,z\,s\,t\,w\,x^2\,y^2 - 2\,z\,s\,t\,w\,y^4$$
$$+ 2\,u^2\,y^4\,z^2 + 2\,u^2\,y^4\,t^2 + u^2\,y^4\,z^4 + u^2\,y^4\,t^4 + 2\,v^2\,x^4\,z^2 + 2\,v^2\,x^4\,t^2 + v^2\,x^4\,z^4 + v^2\,x^4\,t^4$$
$$+ 2\,u^2\,y^2\,x^2\,z^2 + 2\,u^2\,y^2\,x^2\,t^2 + u^2\,y^2\,x^2\,z^4 + u^2\,y^2\,x^2\,t^4 + u^2\,y^2\,x^2 + x^2\,y^2\,v^2 - 4\,u\,y^3\,x\,v\,z^2$$
$$- 4\,u\,y^3\,x\,v\,t^2 - 2\,u\,y^3\,x\,v\,z^4 - 4\,u\,y^3\,x\,v\,z^2\,t^2 - 2\,u\,y^3\,x\,v\,t^4 - 4\,u\,y\,v\,x^3\,z^2 - 4\,u\,y\,v\,x^3\,t^2$$
$$- 2\,u\,y\,v\,x^3\,z^4 - 4\,u\,y\,v\,x^3\,z^2\,t^2 - 2\,u\,y\,v\,x^3\,t^4 + u^2\,y^4 + v^2\,x^4) \,/\, ($$
$$(1 + z^2 + t^2)^2\,(1 + x^2 + y^2)^3))\,(1 + x^2 + y^2)$$

$$\beta := -v\,x + u\,y$$

3d Riemannian

- **Conformal Weyl vanishes**

$$\begin{aligned} R_{jikl} &= + \; Ric_{ik}\, g_{jl} - Ric_{il}\, g_{jk} \\ &\quad + \; g_{ik}\, Ric_{jl} - g_{il}\, Ric_{jk} \\ &\quad - \; \tfrac{1}{2} S \left(g_{ik}\, g_{jl} - g_{il}\, g_{jk} \right) \end{aligned}$$

- **Plug in**

$$Ric_{ij} = \lambda\, g_{ij} = 2K\, g_{ij}$$

- **Get**

$$R_{jikl} = K \left(g_{ik}\, g_{jl} - g_{il}\, g_{jk} \right)$$