Tight closure.

TOP "TEN" LIST of WHY STU

11. Shortens hairy proofs
10. Gives analysts lice?
9. Proves Briançon - Skoda Theorem
8. Proves Ring of invariants are Cohen-Macaulay
7. Proves local homological conjectures
6. Eliminates unsightly cohomology (Jacobian ideal)
5. Makes vanishing theorems
4. Makes contracted extension "bulge" manageble

$I \subseteq R \subseteq S$ (module finite) $\quad IS \cap R \supseteq I$

$\Rightarrow$ splitting theorems

3. Controls your colon ideals

$$(x_1, \dots, x_k) : x_{k+1} \supseteq (x_1, \dots, x_k)$$

where the x's are a system of parameters

2. Compensates failure of regularity
1. If R is regular, it does nothing:

R regular, $I^* = I$

Tight closure in regular ring is a tool for proving elements are in ideals

Let $R = \dfrac{k[x_{ij}]}{I_t(x_{ij})}$ $\quad 1 \le i \le m, \quad 1 \le j \le n$

Every ideal of R is tightly closed, but for $t \ne 1$ R is not regular.

R will be a localization of finitely generated algebra over an excellent ~~ring~~ local ring containing a field.

$x \in I^*$ ($x \in \overline{I}$) $\iff$ this is true mod P for all minimal primes P of R.

So R is a noetherian domain

1. $x \in \overline{I} \iff \forall$ maps $R \xrightarrow{f} V$, V valuation ring
$f(x) \in IV$

if R is noetherian it is enough to look at ~~DVR's~~ V being DVR
when R is a domain it is enough to check $R \hookrightarrow V$.

2. $x \in \overline{I}$ if it satisfies an equation: $x^n + r_1 x^{n-1} + \ldots + r_n = 0$
where the $r_j \in I^j$ ($\Rightarrow x \in \sqrt{I}$)

3. $x \in \overline{I} \iff \exists\, c \ne 0$ in R (R is a domain) s.t. $cx^n \in I^n$ $\forall n \gg 0$
(or $\forall n$
(or for infinitely many n)

1., 2., 3. are definitions of integral closure.

Briançon - Skoda THM

If R is regular equal char 0 and I has m generators then $\overline{I^m} \subseteq I$

if $x^m \in I^m \Rightarrow x \in \overline{I}$ (just take $r_m = -x^m$ and $x^m - r_m = 0$ is the right equation)

then, if

$(u^m, v^m) = I \subseteq R \qquad (u^i v^j)^m = (u^m)^i (v^m)^j \in I^i I^j = I^m, \; i+j=m$

then $\overline{(u^m, v^m)} \supseteq (u^m, u^{m-1}v, u^{m-2}v^2, \dots, v^m)$

if $R = \mathbb{C}\{z_1, \dots, z_n\}$, Ω is a "nice" nbd of the origin

- Briançon - Skoda for R:

  if $f, g_1, \dots, g_k$ are holomorphic function in Ω

  X = common zeros of $g_1, \dots, g_k$

  $$\int_{\Omega - X} \frac{|f|^{\text{certain power}}}{\left(|g_1|^2 + \dots + |g_k|^2\right)^{\text{certain other power}}} [\text{stuff}]\, d\mu < \infty$$

  $\Rightarrow \quad f = \Sigma_j\, g_j h_j \qquad h_j$ holomorphic in Ω

- Hilbert's Nullstellensatz:

  if f vanishes where all the g_i's vanish $\Rightarrow$

  $f \in \operatorname{Rad}(g_1, \dots, g_k)$

R is noetherian, char p

Let $q = p^e$

Def R is a noetherian domain. $x \in R$, $I \subseteq R$ ideal of R.

then $x \in I^*$ if $\exists\, c \in R$ $c \neq 0$ such that

$cx^q \in I^{[q]}$ $\forall$ $q \gg 0$ (all q is equivalent).

where $I^{[q]} = (i^q \mid i \in I) = (f_1^q, \dots, f_k^q)$ where $I = (f_1, \dots f_k)$

$$\downarrow$$

$$(\Sigma_1 x_\nu i_\nu)^q = \Sigma_1 x_\nu^q i_\nu^q$$

$$I^{[q]} \subseteq I^q$$

Easy

1. $I \subseteq I^*$
2. $I \subseteq J \Rightarrow I^* \subseteq J^*$
3. $(I^*)^* = I^*$

If $R \subseteq S$ take $x \in IS \cap R \setminus I$ x is "almost" in I

$\downarrow$ module finite

Let $I = (f_1, \dots, f_k)$

$cx^q = \Sigma_1 x_j f_j^q$

$c^{1/q} x = \Sigma_1 x_j^{1/q} f_j \in IS$, $S = R[c^{\frac{1}{q}}, x_1^{1/q}, \dots, x_k^{\frac{1}{q}}]$

S is module finite over R

Theorem ("Briançon-Skoda")

I m generated : $\overline{I^m} \subseteq I^*$ ($\Rightarrow$ Briançon Skoda for regular rings containing a field of char p).

pf : R domain

$u \in \overline{I^n}$, where $I = (f_1, \dots, f_n)$

$\Rightarrow$ $c \cdot u^m \in (I^n)^m$ $\forall$ m , $c \neq 0$

$\Rightarrow$ $c u^q \in (I^n)^q$ $\forall$ q

$$(I^n)^q = (f_1, \dots f_n)^{nq} \subseteq (f_1^q, \dots, f_n^q) = I^{[q]} . \quad //$$

Also:

<u>THM</u> if R is regular $\Rightarrow I = I^*$ $\forall$ I ideal of R.

pf: suppose R is a domain $c \neq 0$, $c\,m^q \in I^{[q]}$ all q.

Let $F : R \longrightarrow R$, $x \longrightarrow x^p$, $F^e : R \longrightarrow R$, $x \longrightarrow x^{p^e} = x^q$.

if R is regular then F is flat.
(the proof reduces to the local, complete case
$k[|x_1, \dots x_k|] \cong k^p[|x_1^p, \dots x_k^p|] \subseteq k[|x_1, \dots x_k|]$
in this case it is easy to show that the Frobenius map is flat).

<u>Lemma</u> S is flat over R, $I \subseteq R$ $x \in R$

$$IS :_S x = (I :_R x) S$$

pf: exercise. Hint: $\frac{R}{I} \xrightarrow{\cdot x} \frac{R}{I}$ and tensor with S.

back to The Theorem:

$I^{[q]} = IS$ $\quad R \xrightarrow{F^e} R = S$

$I^{[q]} :_R x^q = (I : x)^{[q]}$ if R is regular.

$$c \in \bigcap_q I^{[q]} : x^q = \bigcap_q (I : x)^{[q]} \subseteq \bigcap_q (I : x)^q = 0$$ unless $I : x = R$ i.e. $x \in I$