COMMUTATIVE ALGEBRA IN GROUP COHOMOLOGY

DAVE BENSON
UNIVERSITY OF GEORGIA

I'll discuss cohomology of

- Finite groups
- Finite group schemes
- Compact Lie groups
- p-compact groups
- Infinite discrete groups
- Profinite groups

and the relevance of

- Finite generation
- Maximal / Prime ideal spectrum
- Krull dimension
- Depth
- Associated primes
- Cohen-Macaulay rings and Gorenstein rings
- Local cohomology and Grothendieck's local duality

EXAMPLE

$G = Q_8 \subseteq$ Unit Quaternions $= SU(2)$

acts freely on S^3 by left mult.

S^3/Q_8 is a manifold

so satisfies Poincaré duality

Chains on S^3 give exact sequence

$$0 \to \mathbb{F}_2 \to \text{free} \to \text{free} \to \text{free} \to \text{free} \to \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2 Q_8$ - modules, so $H^*(Q_8, \mathbb{F}_2)$

is periodic, of the form

$$\mathbb{F}_2[z] \otimes H^*(S^3/Q_8, \mathbb{F}_2) \qquad |z| = 4$$

$$H^*(S^3/Q_8, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^2+xy+y^2, x^2y+xy^2)$$

$$|x| = |y| = 1$$

$$\sum_{i=0}^{\infty} t^i \dim_{\mathbb{F}_2} H^i(Q_8, \mathbb{F}_2) = \frac{1+2t+2t^2+t^3}{1-t^4}$$

$\boxed{H^*(Q_8, \mathbb{F}_2) \text{ is a GORENSTEIN ring}}$

Group Cohomology

DEFN: If G discrete group

k commutative ring of coeffs

$$H^*(G,k) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},k) \cong \mathrm{Ext}^*_{kG}(k,k)$$

kG is a Hopf algebra with comultiplication $g \mapsto g \otimes g$ cocommutative. Cup product and Yoneda product agree, to make $H^*(G,k)$ a graded commutative ring

$$ab = (-1)^{|a||b|}\, ba$$

If M is a kG-module then

$$H^*(G,M) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},M) \cong \mathrm{Ext}^*_{kG}(k,M)$$

is a right $H^*(G,k)$-module.

Topological groups

EG = contractible space with free G-action

$BG = EG/G$

$H^*(BG;k)$ = "classifying space cohomology"

Example

$$H^*(BU(n);k) = k[c_1, \dots, c_n]$$

$|c_i| = 2i$ $\quad$ c_i are the Chern classes

If G is discrete then BG is an Eilenberg - Mac Lane space $K(G,1)$. i.e., $\pi_1(BG) = G$, $\pi_i(BG) = 0$ $(i > 1)$. In this case $C_*(EG)$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. So

$$
\begin{aligned}
H^*(BG;k) &= H^* \operatorname{Hom}_{\mathbb{Z}}(C_*(BG), k) \\
&\cong H^* \operatorname{Hom}_{\mathbb{Z}G}(C_*(EG), k) \\
&\cong H^*(G, k).
\end{aligned}
$$

Profinite groups

If $G = \varprojlim_{U \in \mathcal{U}} G/U$, $\bigcap_{U \in \mathcal{U}} U = \{1\}$

and $M = \bigcup_{U \in \mathcal{U}} M^U$ then

<u>continuous cohomology</u> is

$$H^i(G, M) = \varinjlim_{U \in \mathcal{U}} H^i(G/U, M^U).$$

In all the above cases, $H^*(G,k)$ (or $H^*(BG;k)$) is a <u>graded commutative ring</u>

Tate cohomology

G finite, k a field

then injective = projective for

kG-modules. If N is a

kG-module, splice together an

injective and a projective resolution

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \hat P_1 \arrow[r] & \hat P_0 \arrow[rr] \arrow[dr] & & \hat P_{-1} \arrow[r] & \hat P_{-2} \\
& & & N \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

$\Omega^n N$ = nth kernel
$(n \in \mathbb{Z})$

to get a complete resolution

$$\widehat{\mathrm{Ext}}^*_{kG}(N, M) = H^* \operatorname{Hom}_{kG}(\hat P_*, M)$$

$$\hat H^*(G, M) = \widehat{\mathrm{Ext}}^*_{kG}(k, M)$$

$\hat H^*(G,k)$ is a $\mathbb{Z}$-graded comm. ring

Tate duality:

$$\widehat{\mathrm{Ext}}^{n-1}_{kG}(M, k) \cong (\hat H^{-n}(G, M))^*$$

(vector space dual)

FINITE GENERATION

Theorem (Evens) If G finite, k comm. ring of coeffs, and M a kG-module which is Noetherian as a k-module then $H^*(G, M)$ is Noetherian as an $H^*(G, k)$-module.

In particular, if k is Noetherian then $H^*(G, k)$ is a finitely generated k-algebra.

In contrast, $\hat{H}^*(G, k)$ is almost never finitely generated.

GENERALIZATIONS

Theorem (Friedlander-Suslin)

If G is a finite group scheme over a field k (i.e., kG is a f.d. cocomm. Hopf algebra) then $H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ is a finitely generated k-algebra.

If M is a f.g. kG-module then $H^*(G,M)$ is a f.g. $H^*(G,k)$-module.

<u>Theorem</u> (Venkov) G compact Lie, k Noetherian comm. ring.

If $G \to U(n)$ faithful unitary rep, then $H^*(BG;k)$ is f.g. over image of

$$H^*(BU(n);k) = k[c_1, \dots, c_n] \to H^*(BG;k)$$

In particular, $H^*(BG;k)$ is a f.g. k-algebra.

<u>Theorem</u> (Dwyer-Wilkerson)

If $X \simeq \Omega BX$ is a finite loop space (e.g. $G \simeq \Omega BG$) then $H^*(BX;k)$ is a f.g. k-algebra.

<u>Theorem</u> (Dwyer-Wilkerson)

If X is a p-compact group (i.e., X $\mathbb{F}_p$-complete, $\pi_0 X$ p-group, $X \simeq \Omega BX$ and $H^*(X;\mathbb{F}_p)$ finite) then $H^*(BX;\mathbb{F}_p)$ is a f.g. $\mathbb{F}_p$-algebra.

<u>Theorem</u> (Minh - Symonds)

Let G be a pro-p-group.

(i) $H^*(G, \mathbb{F}_p)$ is a f.g. $\mathbb{F}_p$-algebra

$\Leftrightarrow$ G has an open normal torsion-free subgroup U s.t. $H^*(U, \mathbb{F}_p)$ is finite.

(ii) If $H^*(G, \mathbb{F}_p)$ is f.g. then G has a finite number of ccls of finite subgroups.

(iii) $H^*(G, \mathbb{F}_p)$ / nil radical is f.g.

$\Leftrightarrow$ G has a finite number of ccls of elementary abelian p-subgroups (i.e., $(\mathbb{Z}/p)^n$)

<u>Theorem</u> If G is a discrete group with a normal subgroup H such that $\exists$ finite CW-complex BH and k Noetherian comm. ring then $H^*(G;k)$ is Noetherian; i.e., f.g. k-algebra.

<u>Examples</u>

- Arithmetic groups
- Mapping class groups
- Automorphisms of free groups of finite rank

Krull dimension (Quillen)

Def$^{\underline{n}}$ $r_p(G)$ (the p-rank of G) is the largest r for which $(\mathbb{Z}/p)^r \subseteq G$.

If G is a compact Lie group then $r_p(G) \geq r_0(G)$, the Lie rank which is the largest r_0 for which $(S^1)^{r_0} \subseteq G$.

Theorem G compact Lie, k a field of char p ($p=0$ allowed) Then the Krull dimension of $H^*(BG; k)$ is equal to $r_p(G)$.

Theorem The same holds for discrete groups of finite vcd (i.e., $\exists\ H \subseteq G$ finite index s.t. $H^n(H, \mathbb{Z}) = 0$ for n large)

<u>Theorem</u> If G is a profinite group with a finite number of ccls of finite elementary abelian p-subgroups then the same holds.

In these cases, Quillen proved more.

$\mathcal{A}_p(G)$ = category with

objects : $E \cong (\mathbb{Z}/p)^n \leq G$

arrows : group monomorphisms $E \to E'$ induced by conjugation in G

<u>Theorem</u> $H^*(BG;k) \to \varprojlim_{E \in \mathcal{A}_p(G)} H^*(BE;k)$

(restriction) is an <u>inseparable</u> <u>isogeny</u>

i.e., kernel consists of nilpotents & if $x \in$ RHS $\exists\, a \geq 0$ $x^{p^a} \in$ Image.

For an elementary abelian p-group, we have

$$H^*((\mathbb{Z}/p)^r, k) = \begin{cases} k[x_1, \dots, x_r] & |x_i| = 1 \quad (p=2) \\ \Lambda(x_1, \dots, x_r) \otimes k[y_1, \dots, y_r] & \\ |x_i| = 1,\ |y_i| = 2 & (p \text{ odd}) \end{cases}$$

In both cases, mod the nil radical it's a polynomial ring on r generators.

<u>Cor</u> The minimal primes in $H^*(BG; k)$ are in 1-1 correspondence with the ccls of max. elementary abelian p-subgroups of G. If E is a maximal elementary abelian p-subgroup then the corresponding minimal prime is $\sqrt{\mathrm{Ker}\ \mathrm{res}_{G,E}}$.

DEPTH

Harder to compute. Generally only get inequalities.

Sometimes $H^*(G, k)$ is Cohen-Macaulay (depth = Krull dim) e.g. in following cases (k a field):

(i) Groups with abelian Sylow p-subgroups (Duflot)

(ii) $GL(n, \mathbb{F}_q)$, char $k \nmid q$ (Quillen)

& Other finite groups of Lie type away from defining characteristic

(Fiedorowicz & Priddy for classical gps, Kleinerman for exceptional Lie type)

(iii) Extraspecial 2-groups (Quillen)

(iv) Finite simple groups of 2-rank ≤ 3 with char $k = 2$ (Adem & Milgram)

<u>Theorem</u> (Duflot, ...)

Let G be a compact Lie group and let S be a Sylow p-toral subgroup of G (i.e., take a maximal torus T and let S be the inverse image in G of a Sylow p-subgroup of $N_G(T)/T$)

Then
$$\operatorname{depth} H^*(BG;k) \geq r_p(Z(S))$$

In particular, if $r_p(G) \geq 1$ then $r_p(Z(S)) \geq 1$ so $H^*(BG;k)$ has depth ≥ 1. i.e., it has a non zero-divisor in positive degree.

David Green strengthened this to:

<u>Theorem</u> If there is a regular sequence of length s consisting of <u>primitive</u> <u>elements</u> for the coaction of $H^*(BZ(S);k)$ then

$$\operatorname{depth} H^*(BG;k) \geqslant s + r_p(Z(S))$$

Here, $\mu : Z(S) \times S \to S$ (mult.) induces

$$\mu^* : H^*(BS;k) \to H^*(BZ(S);k) \otimes H^*(BS;k)$$

so $H^*(BS;k)$ is an $H^*(BZ(S);k)$-comodule algebra

$H^*(BG;k)$ is a sub-comodule algebra

and x is <u>primitive</u> if

$$\mu^*(x) = 1 \otimes x$$

e.g. Inflations from $H^*(BS/Z(S);k)$ to $H^*(BG;k)$ are primitive.