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Tight closure. 2.

How do you know $x \in I^*$?

This is an hard problem because you need to check infinitely many equations $cx^{p^e} \in I^{[p^e]}$ for infinitely many e.

Example:

$$\frac{k[x,y,z]}{(x^3+y^3+z^3)} \qquad \text{char } k \neq 3$$

we want to show $z^2 \in (x,y)^*$ but $z^2 \notin (x,y)$ ($z^2 \neq 0$ in

$$\frac{k[x,y,z]}{(x,y,x^3+y^3+z^3)} \cong \frac{k[z]}{z^3})$$

For the $z^2 \in (x,y)^*$ I need to check the following:

CLAIM

$$x(z^2)^q \in (x,y)^{[q]} = (x^q, y^q)$$

$q = a \bmod 3$, $q=1$ or $q=2$

$2q = 6h + 2a$ since we can write $q = 3h + a$

$x z^{2q} = x z^{6h+2a} = \pm x (x^3+y^3)^h z^b$ rewrite $q = 3h + b$

$$= \pm x \left(\sum_i \binom{h}{i} x^{3i} y^{3j} \right) z^b$$

either $3i + 1 \geq q$

or $3j \geq q$

if $3i + 1 \leq q-1$

and $3j \leq q-1$

$\Rightarrow 3h + 1 \leq 2q - 2 \Rightarrow 2q \geq 3h + 3$ *

so $x \cdot (z^2)^q \in (x^q, y^q) \quad \forall q \gg 0$

Testing ~~either~~ whether $u \in I^*$ in $\dfrac{k[x,y,z]}{(x^3+y^3+z^3)}$

is a very hard problem:

if R is a finitely generated $\mathbb{Q}$-algebra and $I \subseteq R$

how do you test tight closure?

Find a finitely generated $\mathbb{Z}$-algebra $A \subseteq R$ s.t. $x \in A$,

$J = I \cap A$ generates I, such that $\mathbb{Q} \otimes_{\mathbb{Z}} A = R$

(example $\mathbb{Q} \otimes \dfrac{\mathbb{Z}[x,y,z]}{(x^3+y^3+z^3)} = \dfrac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}$).

Def define $x \in I^*$ if for almost all primes p the image of x is in $(JA/pA)^*$ (in the char p sense)

(Use locally excellent rings that contains $\mathbb{Q}$)

Example:

$$\text{in } \frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)} \longrightarrow \frac{\mathbb{Z}_p[x,y,z]}{(x^3+y^3+z^3)}$$

we just checked in the previous example that

$z^2 \in (x,y)^*$, in the example almost all q

where fine.

the definition is independent by the choice of A.

going back to char p:

if $x \in I^*$, $I \subseteq R$ ideal

$R \xrightarrow{f} S$ any ring homomorphism

$f(x) \in (IS)^*$ over S.

This is called Persistence. the proof is not easy.

Def of tight closure for locally excellent Noeth. rings containing $\mathbb{Q}$.

$x \in R, I \subseteq R$

$x \in I^*$ if for all maps $R \xrightarrow{f} B$, B is a complete local domain.

$\exists$ a finitely generated $\mathbb{Q}$-algebra T and a map $T \to R$ and $x_0 \in T$, $I_0 \subseteq T$

s.t. $x_0 \in I_0^*$ in T and $x_0 \mapsto f(x)$ in B, and $I_0 B = IB$.

there are a lot of open questions about the char 0 def.

THM (Ein-Lazarsfeld-Smith comparison thm)

$R \supseteq$ field. P is an height h prime of R, R is regular.

Then:

$$P^{(hn)} \subseteq P^n \quad \forall n.$$

where $$P^{(n)} = P^n R_P \cap R$$

This is also true if R/P has finite projective dim.

(hint: $\text{pd}\, R/P < \infty \Rightarrow P^{[q]}$ is unmixed)

THM (Hochster - Roberts)

if R is a noetherian k-algebra, regular. G is a linearly reductive algebra ~~gp~~ groups over k, acting over $R \Rightarrow R^G$ is Cohen Macaulay

$\downarrow$

ring of invariants.

There is an R^G-linear map from R to R^G i.e

$R^G \hookrightarrow R$ splits as a map of R^G modules.

THM: (Hochster–Huneke)
if S is regular, $R \to S$ splits as a map of R-modules
$\Rightarrow$ R is Cohen–Macaulay
(the proofs reduces to the case where R is local and complete).

• Colon capturing:

THM If R is a local domain "nice" (for example excellent)
and let $x_1, \dots x_{k+1}$ part of a system of parameters:

$$(x_1, \dots x_k) : x_{k+1} \subseteq (x_1, \dots, x_k)^*$$

pf: Let R be a complete local, domain. Let
k be a coefficient field of R then:

$$k[|x_1, \dots, x_k, \dots x_m|] = A$$

$$A \subseteq R$$
module finite

$u \cdot x_{k+1} \in (x_1, \dots, x_k)$ claim: $u \in (x_1, \dots x_k)^*$

choose $A^h \subseteq R$ free and pick $c \in A - \{0\}$ s.t.
$c \cdot R \subseteq A^h$

$$(u^q \cdot x_{k+1}^q) \in (x_1^q, \dots, x_k^q)R$$

$$c u^q x_{k+1}^q \in (x_1^q, \dots, x_k^q)A^h$$

$$\Rightarrow c u^q \in (x_1^q, \dots x_k^q)A^h \quad . \text{ but } A^h \subseteq R$$

$$\Rightarrow c u^q \in (x_1^q, \dots x_k^q)R \quad \text{all } q.$$

THM R is a direct summand of S, S regular. $\Rightarrow$
R is Cohen Macaulay

pf: R complete local., $S = R \oplus W$ as R-modules.

if $u \in I^*$ in R, $u \in (IS)^* = IS$. $u \in R \cap IS$

$$R \cap IS = R \oplus 0 \cap (I \oplus IW) = I$$

in R $I = I^*$ for all I

$(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^* = (x_1, \ldots, x_k)$ //