Maschke's Theorem

If $\mathrm{char}\, k \nmid |G|$ then every short exact sequence of kG-modules <u>splits</u>

so $\mathrm{Ext}^{>0}_{kG}(-,-) = 0$

In particular $H^{>0}(G,k) = 0$.

COMPUTING COHOMOLOGY OF A FINITE GROUP

FACT $|G|$ annihilates $H^{>0}(G,-)$ so only need to compute p-torsion for each $p \mid |G|$.

TRANSFER MAP tells us that if S is a Sylow p-subgroup of G then

$$\mathrm{res}_{G,S} : H^*(G,k) \to H^*(S,k)$$

is injective.

CARTAN - EILENBERG STABLE ELEMENT METHOD

The image of $\mathrm{res}_{G,S}$ consists of elements such that if $g \in G$ & $H, H^g \subseteq S$ then

$$\mathrm{conj}_g \circ \mathrm{res}_{S,H} = \mathrm{res}_{S,H^g}$$

"Stable elements w.r.t. fusion in G"

If G is a p-group,

BY HAND 1) Do cyclic groups using explicit resolutions

2) Use induction on order. Choose a central cyclic subgroup $1 \neq Z \leq G$ and use either the LHS Spectral sequence

$$H^*(G/Z, H^*(Z,k)) \Rightarrow H^*(G,k)$$

or the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{**}^{H^*(K(Z,2),k)}(k, H^*(G/Z,k)) \Rightarrow H^*(G,k)$$

The latter was used by Rusin to compute mod 2 cohomology of groups of order 32; the former was used by Quillen to compute cohomology of extraspecial 2-groups.

MACHINE COMPUTATION (Carlson)

Compute an explicit free resolution of k as kG-module. Don't build the modules, just write the maps as matrices with entries in kG

Calculating when to stop resolving: use some commutative algebra! (see Lecture 3)

Carlson computed the mod 2 cohomology of the groups of order 64 this way.

<u>Example</u> Normal abelian Sylow p-subgroups in char p

$$1 \to A \to G \to G/A \to 1$$

LHS Spectral Sequence collapses to give

$$H^*(G, k) = H^*(A, k)^{G/A}$$

The inclusion $H^*(G,k) \hookrightarrow H^*(A,k)$ <u>splits</u> (as an $H^*(G,k)$-module) so $H^*(G,k)$ is Cohen-Macaulay

Associated Primes & Steenrod Operations

Group cohomology with $\mathbb{F}_p$ coeffs admits an action of the Steenrod operations $(i \geq 0)$

$$Sq^i : H^n(BG;\mathbb{F}_2) \to H^{n+i}(BG;\mathbb{F}_2) \quad (p=2)$$

$$P^i : H^n(BG;\mathbb{F}_p) \to H^{n+2i(p-1)}(BG;\mathbb{F}_p) \quad (p \text{ odd})$$

These satisfy "Adem relations"

Steenrod algebra is generated by Steenrod operations, modulo Adem rels.

As an algebra over the Steenrod algebra $H^*(BG;\mathbb{F}_p)$ contains a copy of the Dickson invariants as a h.s.o.p.

$$\mathbb{F}_p[x_1,\dots,x_r]^{GL_r(\mathbb{F}_p)} = \mathbb{F}_p[c_{r,r-1},\dots,c_{r,0}]$$

dilated via $|x_i| = 2p^a$,

$$|c_{r,i}| = 2(p^{a+r} - p^{a+i}).$$

The Landweber - Stong conjecture proved by Bourguiba & Zarati says that for any unstable module over the Steenrod algebra, which is simultaneously a module over the Dickson invariants with a suitable compatibility condition (for example $H^*(BG; \mathbb{F}_p)$) the depth is the largest d for which $c_{r,r-1}, \ldots, c_{r,r-d}$ is a regular sequence.

i.e., we can use a single test sequence!

Conjecture (Carlson) Let G be a finite group and k a field. Then $H^*(G,k)$ has an associated prime whose dimension is the depth

Theorem (Wilkerson)

If H is an unstable algebra over the Steenrod algebra (i.e., $Sq^i(x) = 0$ for $i > |x|$ $(p=2)$
$P^i(x) = 0$ for $i > 2(p-1)|x|$ $(p$ odd$)$)

e.g. mod p cohomology of a space

Then the radical of the annihilator of any element is Steenrod invariant.

In particular,

associated primes are Steenrod invariant

Theorem (Serre)

Let E be an elementary abelian p-group. Then the Steenrod invariant prime ideals in $H^*(E, \mathbb{F}_p)$ are in 1-1 corr. with the subgroups of E.

If $E' \subseteq E$ then the corresponding Steenrod invariant prime ideal is

$$\sqrt{\operatorname{Ker} \operatorname{res}_{E,E'}} .$$

Combining with Quillen's inseparable isogeny, we get:

__Theorem__ G compact Lie

The Steenrod invariant prime ideals in $H^*(BG;\mathbb{F}_p)$ are the ideals of the form

$$\sqrt{\operatorname{Ker} \operatorname{res}_{G,E}}$$

where E is an elementary abelian p-subgroup of G.

In particular, the associated primes are of this form, but for which elementary abelian p-subgroups E?

– Difficult question.

IDEMPOTENT MODULES AND VARIETIES

G finite group, char $k = p$

V_G = max ideal spectrum of $H^*(G,k)$

e.g. If $G \cong (\mathbb{Z}/p)^r$ then $V_G \cong \mathbb{A}^r(k)$

Quillen's inseparable isogeny gives

$$\varinjlim_{E \in \mathcal{A}_p(G)} V_E \longrightarrow V_G$$

is bijective at level of points (usually not invertible in category of varieties)

(Carlson)

If M is a finitely generated kG-module, the kernel of

$$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k) \xrightarrow{\cdot \otimes M} \mathrm{Ext}^*_{kG}(M,M)$$

is an ideal in $H^*(G,k)$ which defines a closed homogeneous subvariety $V_G(M) \subseteq V_G$

Get same answer from the ideal

$$\bigcap_{S \text{ simple}} \text{Annihilator of } \mathrm{Ext}^*_{kG}(S,M)$$

Properties of $V_G(M)$

(i) $V_G(M) = \{0\} \Leftrightarrow M$ is projective

(ii) $V_G(M \oplus N) = V_G(M) \cup V_G(M)$

(iii) $V_G(M \otimes_k N) = V_G(M) \cap V_G(N)$

(iv) If $0 \neq \zeta \in H^*(G,k)$ is represented by $\hat{\zeta} : \Omega^n k \to k$ with kernel L_ζ then $V_G(L_\zeta)$ is the hypersurface determined by ζ in V_G, denoted $V_G\langle\zeta\rangle$

For any closed hgs $V \subseteq V_G$ can choose $\zeta_1, \dots, \zeta_t$ with

$$V = V_G\langle\zeta_1\rangle \cap \dots \cap V_G\langle\zeta_t\rangle$$
$$= V_G(L_{\zeta_1}) \cap \dots \cap V_G(L_{\zeta_t})$$
$$= V_G(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_t}).$$

INFINITE DIMENSIONAL MODULES

Why?

1. Various constructions naturally give infinite dimensional modules
2. Representing objects for functors are often infinite dim.

Analogy

In algebraic topology, even if you're only interested in finite CW complexes, cohomology is represented by Eilenberg-Mac Lane spaces, K-theory by BU, cobordism by MU, etc. These are all infinite complexes.

$\mathrm{Mod}(kG)$ = category of <u>all</u> kG-modules and homomorphisms

$\mathrm{StMod}(kG)$: same objects, morphisms

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

$\mathrm{PHom}_{kG}(M,N)$ = subspace of maps which factor through a projective kG-module

$\mathrm{StMod}(kG)$ is <u>not</u> an abelian category — can't tell whether a map is injective or surjective

Instead it's <u>triangulated</u>

Triangle $A \to B \to C \to \Omega^{-1} A$ corresponds to a short exact sequence

$$0 \to A \to B \oplus (\text{proj}) \to C \to 0$$

$\mathrm{mod}(kG)$, $\mathrm{stmod}(kG)$ full subcategories of f.g. modules

$\operatorname{Proj} H^*(G,k)$ = set of closed homogeneous irreducible subvarieties of V_G.

If $\mathcal{V} \subseteq \operatorname{Proj} H^*(G,k)$ is closed under specialization (i.e., if $V \in \mathcal{V}$, $W \subseteq V$ then $W \in \mathcal{V}$)

$\mathcal{M}$ = full subcategory of $\operatorname{stmod}(kG)$ consisting of modules M s.t. $V_G(M)$ is a finite union of elements of $\mathcal{V}$

then $\mathcal{M}$ is a <u>thick</u> <u>subcategory</u> of $\operatorname{stmod}(kG)$, <u>ideal</u> <u>closed</u>

To such a subcategory, Rickard associates a triangle

$$E_{\mathcal{V}} \to k \to F_{\mathcal{V}} \to \Omega^{-1} E_{\mathcal{V}}$$

of idempotent modules

Characterized by:

$E_{\mathcal{V}}$ is a filtered colimit of modules in $\mathcal{M}$

For any M in $\mathcal{M}$, we have $\underline{\mathrm{Hom}}_{kG}(M, F_{\mathcal{V}}) = 0$

<u>Example</u> If $V = V_G\langle \zeta \rangle$ $\zeta \in H^{n}(G,k)$ & $\mathcal{V}$ = set of subvarieties of V

write E_ζ, F_ζ for $E_{\mathcal{V}}$, $F_{\mathcal{V}}$

Represent ζ by a cocycle $\hat{\zeta} : \Omega^n k \to k$ and dimension shift:

$$k \xrightarrow{\Omega^{-n}\hat{\zeta}} \Omega^{-n} k \xrightarrow{\Omega^{-2n}\hat{\zeta}} \Omega^{-2n} k \to \cdots$$

colimit is F_ζ

$$\hat{H}^*(G, F_\zeta) \cong \hat{H}^*(G,k)_\zeta \cong H^*(G,k)_\zeta$$

Get E_ζ from completing $k \to F_\zeta$ to a triangle

$$\begin{tikzcd}
\Omega^{-n} L_\zeta \arrow[r] \arrow[d] & k \arrow[r, "\Omega^{-n}\hat{\zeta}"] \arrow[d, equal] & \Omega^{-n} k \arrow[d, "\Omega^{-2n}\hat{\zeta}"] \\
\Omega^{-2n} L_{\zeta^2} \arrow[r] \arrow[d] & k \arrow[r, "\Omega^{-2n}\widehat{\zeta^2}"] \arrow[d, equal] & \Omega^{-2n} k \arrow[d, "\Omega^{-3n}\hat{\zeta}"] \\
\Omega^{-3n} L_{\zeta^3} \arrow[r] \arrow[d] & k \arrow[r, "\Omega^{-3n}\widehat{\zeta^3}"] \arrow[d, equal] & \Omega^{-3n} k \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

$\varinjlim$ gives

$$E_\zeta \to k \longrightarrow F_\zeta$$

If $V \subseteq V_G$ is defined by $\zeta_1, \dots, \zeta_t$
and $\mathcal{V} = \{$ subvarieties of $V \}$
write E_V for $E_{\mathcal{V}}$, F_V for $F_{\mathcal{V}}$

$$E_V = E_{\zeta_1} \otimes \cdots \otimes E_{\zeta_t}$$

Complete $E_V \to k$ to a triangle
to get F_V.

If V closed hgs irred $\subseteq V_G$
set $\mathcal{W} = \{$ subvarieties not containing $V \}$

$$\kappa_V = E_V \otimes F_{\mathcal{W}}$$

is an idempotent module
corresponding to "the generic
point of V" — all proper
subvarieties have been removed.

For any M in $\mathrm{StMod}(kG)$

$$\mathcal{V}_G(M) = \{ V \mid M \otimes_k \kappa_V \text{ not proj} \}$$
$$\subseteq \operatorname{Proj} H^*(G,k)$$

- If M f.g. then $\mathcal{V}_G(M)$ is the set of subvarieties of $V_G(M)$
- $\mathcal{V}_G(M) = \emptyset \iff M$ is projective
- $\mathcal{V}_G(M \oplus N) = \mathcal{V}_G(M) \cup \mathcal{V}_G(N)$
- $\mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$
- $\mathcal{V}_G(\kappa_V) = \{V\}$

In ptic, every subset of $\operatorname{Proj} H^*(G,k)$ is $\mathcal{V}_G(M)$ for some M.

DUALITY THEOREMS

Original Poincaré duality spectral sequence : B-Carlson (1994) based on using L_ζ's to make multiple complexes of projective modules.

Easier version to work with: Greenlees spectral sequence (1995)

$$H^{s,t}_{\mathfrak{m}} H^*(G,k) \Rightarrow H_{-s-t}(G,k)$$

Example of consequences:

1. If $H^*(G,k)$ is Cohen-Macaulay then it is Gorenstein.
2. If $\mathfrak{p}$ is a minimal prime then $H^*(G,k)_{\mathfrak{p}}$ is Gorenstein i.e., $H^*(G,k)$ is <u>generically</u> Gorenstein

CONSTRUCTION

Choose a h.s.o.p.

$$k[\zeta_1, \dots, \zeta_r] \overset{\text{f.g.}}{\subseteq} H^*(G,k)$$

For each ζ_i we have a triangle

$$E_{\zeta_i} \to k \to F_{\zeta_i} \to \Omega^{-1} E_{\zeta_i}$$

Truncate to make a complex

$$\dots \to 0 \to \underset{0}{k} \to \underset{1}{F_{\zeta_i}} \to 0 \to \dots$$

with cohomology $\Omega^{-1} E_{\zeta_i}$ in deg 1.

Tensor these together:

$$\Lambda^* : 0 \to k \to \bigoplus_{1 \le i \le r} F_{\zeta_i} \to \dots \to \bigotimes_{1 \le i \le r} F_{\zeta_i} \to 0$$

cohomology is

$$\Omega^{-1} E_{\zeta_1} \otimes \dots \otimes \Omega^{-1} E_{\zeta_r}$$

$$\mathcal{V}_G(\Omega^{-1}E_{\zeta_1} \otimes \cdots \otimes \Omega^{-1}E_{\zeta_r})$$
$$= \mathcal{V}_G(\Omega^{-1}E_{\zeta_1}) \cap \cdots \cap \mathcal{V}_G(\Omega^{-1}E_{\zeta_r})$$
$$= \bigcap_{i=1}^{r} \{\text{subvarieties of } V_G\langle \zeta_i \rangle\}$$
$$= \phi$$

so $\Omega^{-1}E_{\zeta_1} \otimes \cdots \otimes \Omega^{-1}E_{\zeta_r}$ is <u>projective</u>

Let $\hat{P}_*$ be a Tate resolution of k as a kG-module. Form a double complex

$$\hat{E}_0^{**} = \mathrm{Hom}_{kG}(\hat{P}_*, \Lambda^*)$$

Differential from Λ^* first:

E_1 is $\mathrm{Hom}_{kG}(\hat{P}_*, H^*(\Lambda))$

projective

so $\underline{\underline{E_2 = 0}}$

So the other spectral sequence must also $\Rightarrow 0$.

Differential from $\hat{P}_*$ first:

$$\hat{E}_1^{**} = C^*(\hat{H}^*(G,k) : \zeta_1, \ldots, \zeta_r)$$

(because $\hat{H}^*(G, F_\zeta) \cong H^*(G,k)_\zeta$
$\hat{H}^*(G,k)_\zeta$)

— stable Koszul complex for computing local cohomology

$$\boxed{\hat{E}_2^{**} = H_{m}^{s,t}\, \hat{H}^*(G,k) \Rightarrow 0}$$

Almost, but not quite, Greenlees spectral sequence

E_0^{**} = subcomplex of $\hat{E}_0^{**}$ consisting of all but the terms $\hat{E}_0^{0,t}$ with $t<0$.

$$0 \to \mathrm{Tot}\, E_0^{**} \to \mathrm{Tot}\, \hat{E}_0^{**} \to \mathrm{Hom}_{kG}(\hat{P}_*^-, k) \to 0$$

$$\begin{aligned} H^n \,\mathrm{Tot}\, E_0^{**} &\cong H^{n+1} \mathrm{Hom}_{kG}(\hat{P}_*^-, k) \\ &\cong H_{-n}(G, k) \end{aligned}$$

(Tate duality)

$$E_1^{**} = C^*(H^*(G,k) ; \zeta_1, \ldots, \zeta_r)$$

$$E_2^{s,t} = H_{\mathfrak{m}}^{s,t} H^*(G,k) \Rightarrow H_{-s-t}(G,k)$$

<u>This</u> is the Greenlees spectral sequence.

<u>Rewrite</u>:

$$H_{\mathfrak{m}}^{**} H^*(G,k) \Rightarrow I_{\mathfrak{m}}$$

<u>Remark</u> Only has nonzero columns between depth & Krull dimension.

Remark

If we remove the *whole* $s=0$ line, not just the negative part, we obtain the *Čech complex* for $H^*(G,k)$:

$$\check{H}^{**}_{\mathfrak{m}}\, H^*(G,k) \Rightarrow \hat{H}^*(G,k)$$

Theorem Let G be a finite group of p-rank r and k be a field of char p. If $H^*(G,k)$ is Cohen-Macaulay, then it is Gorenstein, with canonical module $H^*(G,k)[-r]$.

Proof In the Cohen-Macaulay case, the Greenlees spectral sequence has only one column non-vanishing.