Melvim Hochster.

Tight closure 3

<u>Def</u> weekly f- regular are rings where all the ideals are tightly closed.

Weekly f-regular $\Rightarrow$ cohen macaulay, normal

if S is module finite over R $\qquad IS \cap R \subseteq I^*$

$\mathbb{Q} \subseteq R$ and R normal

then $IS \cap R = I$ automatically.

$R \subseteq S$ module finite, R reduced and excellent

then $IS \cap R = I \ \forall I \iff R \hookrightarrow S$ splits over R.

Let R a domain:

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \text{normal} \cap & & \cap \\ K & & \mathcal{L} \end{array} \quad \text{s.t. } [\mathcal{L} : K] = d$$

$$\frac{1}{d} \operatorname{Tr}_{\mathcal{L}/K} : S \longrightarrow R \text{ is an } R\text{-module retraction.}$$

EX:

$$\frac{k[x,y,z]}{(x^3+y^3+z^3)} \qquad \operatorname{char} k = 2$$

$$z^2 \in (x,y)^*$$

$$z^4 = (x^3+y^3)z$$

$$z^2 = (x\sqrt{x} + y\sqrt{y})\sqrt{z} \in (x,y).$$

Let R^+ = integral closure of R in an algebraical closure of ~~a~~ fractional ~~[illegible]~~ field of R

$I^+ = IR^+ \cap R$

Thm : $I \subseteq I^+ \subseteq I^*$

is an open question if $I^+ = I^*$ (R locally excellent domain).

if $I = I^*$ all $I \Rightarrow$ R is a direct summand of every module - finite extention.

$IS \cap R \subseteq I^* = I$.

Corollary Regular ring of char p are direct summands of all module - finite extentions

Heitmann : if $\dim R = 3$, R regular, local of mixed characteristic $\Rightarrow$ R is a direct summand of all module - finite extentions.

Key - lemma :
R, x, y, z are a sop. p is mixed characteristic.

$ux + vy + wz = 0 \Rightarrow p^{\frac{1}{N}} w \in (x, y)R^+ \ \forall N$.

Karen Smith : if I is generated by a system of parameters then $IR^+ \cap R = I^*$.

THM R is a complete local domain, then $x \in I^* \Leftrightarrow$ $x \in IB$, where B is a big Cohen Macaulay algebra for R, for some B.

Big Cohen Macaulay means: $m_R B \neq B$

and sop for R are regular in B.

THM R^+ is a big Cohen Macaulay algebra, char p.

If R is reduced, "nice". (finite type over an excellent ring), Then

$\bigcap_{I \subseteq R} \operatorname{ann}_R I^*/I =: \tau_R \neq 0$ $\qquad$ τ_R: are the all Test elements.

If R is a domain if $c \in \tau_R$, $c \neq 0$ then $x \in I^*$

$\iff cx^q \in I^{[q]}$ for all q.

More is True:

if k is algebraically closed, R is a domain ~~over~~ f.g.

over ~~R~~ k, then the Jacobian ideal $\subseteq \tau_R$.

(This uses the Lipman Sathaye Jacobian theorem).

EX: $\dfrac{k[x,y,z]}{(x^3+y^3+z^3)}$ char $k \neq 3$ then $3x^2, 3z^2, 3y^2$ are

test elements but there are more.

Tight closure for modules:

if G is a free R-module (maybe infinitely generated)

Let ~~g~~ choose a basis.

$g \in G$ $\quad g^{p^e}$ = raise the coefficients to the p^e power

of course it depends from the choice of the basis.

R is a domain, $I \subseteq G$ submodule

$$I^{p^e} = R\text{-span} \langle i^{p^e} \mid i \in I \rangle$$

$$u \in I^* \text{ if } \exists\, c \neq 0 \text{ s.t. } cu^{p^e} \in I^{[p^e]} \quad \forall p \gg 0$$

if $N \subseteq M$ and $u \in M$. to test $u \in N^*$, map a free module $G \xrightarrow[f]{} \!\!\!\!\rightarrow M$, pick $g \in G$ such that $f(g) = u$. Let $I = f^{-1}(N)$, $u \in N^*_M$ if $g \in I^*_G$.

From now on we'll consider finitely generated modules.

In a complex:

$$G_{i+1} \xrightarrow{d_{i+1}} G_i \xrightarrow{d_i} G_{i-1}$$

if $u \in \ker d_i = Z_i$

if $u \in \operatorname{Im} d_{i+1} = B_i$

then $\bar{u} = 0$ in homology.

if $u \in B^*_{i\,G_i}$ we call $\bar{u} \in H_i(G_\cdot)$ a <u>phantom element</u>. ($u \in Z_i$)

Suppose

$0 \to G_m \xrightarrow{d_m} \dots \xrightarrow{d_1} G_0 \to 0$ finite free complex.

acyclicity if:

$rk\, G_i = b_i$ $\qquad G_i \cong R^{b_i}$

$rk\, d_i = r_i$

$b_i = r_{i+1} + r_i$

depth $I_{r_i}(d_i) \geq i$, $1 \leq i \leq m$

if R is a domain ("nice"), given a complex $0 \to G_m \to \dots \to G_0 \to 0$, then if the same rank condition + height $I_{r_i}(d_i) \geq i$, all i

$\Rightarrow H_i(G_\cdot)$ is phantom $\forall i \geq 1$.

Two application:

THM let $A \subseteq R \longrightarrow S$, A is regular, R is module finite over A

S is regular.

Then, if M is any A-module (you can assume it's f.g),

$$\mathrm{Tor}_i^A(M,R) \longrightarrow \mathrm{Tor}_i^A(M,S) \quad i \geq 1.$$

the map is zero!

(This is not know in mixed characteristic)
(known in char p or equicharacteristic 0)

Sketch:

$$G_\bullet \qquad \underbrace{0 \to A^{b_m} \to \cdots \to A^{b_0}}_{\text{satisfies Buch.-Eis. criteria.}} \to M \to 0$$

Tensor with R, satisfies phantom acyclicity over R.

$R \otimes G_\bullet$ has phantom homology.

$$R \otimes G_\bullet \longrightarrow G_\bullet \otimes S$$

phantom elements $\longrightarrow 0$ in regular enviroment
(S is regular)

//

Say that R is local, $x_1, \dots, x_m$ are a sop.

Koszul complex has phantom homology

$\Rightarrow$ Jacobian ideal annihilates koszul cohomology $\Rightarrow$ the same for $H^i_m(R)$ for $i < m$

The same happens if R is graded $X = \mathrm{Proj}\, R$ on

$\bigoplus_t H^i(X, \mathcal{O}_X(t)) \quad i \geq 2.$