Lazarsfeld B

Ruchira Datta MSRI Comm. Alg. Intro. Workshop Notes

Invariants from Multiplier Ideals

Recall: X = smooth complex affine variety (e.g., $\mathbb{C}^d$)
$\mathfrak{a} \subseteq \mathbb{C}[X]$, $c > 0$ rational
$\mathcal{J}(\mathfrak{a}^c) \subseteq \mathbb{C}[X]$

Def Log-canonical threshold (lct) of $\mathfrak{a}$ is

$$\mathrm{lct}(\mathfrak{a}) = \min\{c > 0 \mid \mathcal{J}(\mathfrak{a}^c) \subsetneq \mathbb{C}[X]\}$$

Intuition: small lct $\rightsquigarrow$ nasty singularities of $f \in \mathfrak{a}$

Recall: Examples
$\mathrm{lct}(\mathfrak{a})$ ~~for~~: $\mathfrak{a} = (s^2, t^2)$: 1
(s^3, t^2) : $5/6 < 1$

If $\mathfrak{a} = (t_1^{m_1}, \dots, t_d^{m_d})$, $\mathrm{lct}(\mathfrak{a}) = \sum 1/m_i$

Proposition/Definition Given $\mathfrak{a} \subseteq \mathbb{C}[X]$,
$\exists$ increasing sequence of rational numbers
$0 = \xi_0 < \xi_1 < \xi_2 < \cdots$
s.t. $\mathcal{J}(\mathfrak{a}^c)$ are constant for $c \in [\xi_i, \xi_{i+1})$
$\xi_i = \xi_i(\mathfrak{a})$ are the jumping numbers of $\mathfrak{a}$

NB: $\xi_1(\mathfrak{a}) = \mathrm{lct}(\mathfrak{a})$

Example $\mathfrak{a} = (t_1^{m_1}, \dots, t_d^{m_d}) \subseteq \mathbb{C}[t_1, \dots, t_d]$
Then $t_1^{e_1} \cdots t_d^{e_d} \in \mathcal{J}(\mathfrak{a}^c)$ iff $c < \sum \frac{e_i + 1}{m_i}$

Jumping numbers of $\mathfrak{a}$ are $\left\{\sum \frac{e_i+1}{m_i}\right\}_{e_1, \dots, e_d \in \mathbb{N}}$

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Take $f \in \mathbb{C}[X]$
Then $\mathcal{J}((f)) = (f)$
$\mathcal{J}((f)^c) \overset{def}{=} \mathcal{J}(f^c) \supsetneqq (f)$ if $c < 1$
$\Rightarrow \xi = 1$ is a jumping number for f (i.e., (f)),
say $\xi_\ell(f) = 1$

Def $\ell = \ell(f)$ is the jumping length of f

(counts # of jumping numbers ≤ 1)

Example: $f = s^4 + t^3 \in \mathbb{C}[s,t]$
f is sufficiently generic so that for $c<1$,
can compute $\mathcal{J}(f^c)$ in terms of (s^4, t^3)

$$0 < \underset{\xi_1}{\frac{1}{4}+\frac{1}{3}} < \underset{\xi_2}{\frac{2}{4}+\frac{1}{3}} < \underset{\xi_3}{\frac{1}{4}+\frac{2}{3}} < \underset{\xi_4}{1} \Rightarrow \ell(f) = 4$$

Example $(f=0)$ nonsingular
then $\ell(f) = 1$

Uniform Artin-Rees

Consider $f \in \mathbb{C}[X]$, $\mathfrak{b} \subseteq \mathbb{C}[X]$
Artin-Rees: $\exists\, k = k(f, \mathfrak{b})$ s.t.
$(*)$ $\mathfrak{b}^m \cap (f) \subseteq (f)\, \mathfrak{b}^{m-k}$, $m \geq k$
i.e., $fg \in \mathfrak{b}^m \Rightarrow g \in \mathfrak{b}^{m-k}$
Huneke: can take $k = k(f)$ indep of $\mathfrak{b}$
(we say k is a uniform Artin-Rees number of f)
Q: What geometric information does the uniform Artin-Rees number of f depend on?

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Theorem (Ein, Laz, Smith, Varolin)
The integer $k = \ell'(f)\dim(X)$
is a UAR (uniform Artin-Rees) number for f.

Example ~~...~~ $(f=0)$ smooth.
$\ell(f) = 1$, so $\dim(X)$ is UAR number.

Prop/Cor Assume $(f=0)$ has an isolated singularity at $x \in X$, otherwise nonsingular.
Then $\ell(f) \leq \cancel{\mu_x(f)+1}\ \tau_x(f)+1$
$\tau_x(f)$ Turina number
= colength $(f, \partial f/\partial z_i)$

Setup of Proof:

Consider consecutive jumping numbers
$\xi' = \xi_i(f) < \xi_{i+1}(f) = \xi$
so $\mathcal{J}(f^{\xi'}) \supsetneq \mathcal{J}(f^{\xi})$

Main Lemma For any $b \subseteq \mathbb{C}[X]$

$$b^m \cdot \mathcal{J}(f^{\xi'}) \cap \mathcal{J}(f^{\xi}) \subseteq b^{m-d}\, \mathcal{J}(f^{\xi})$$

(Apply to each link in
$(1) = \mathcal{J}(f^0) \supseteq \mathcal{J}(f^{\xi_1}) \supseteq \mathcal{J}(f^{\xi_2}) \supseteq \cdots \supseteq \mathcal{J}(f) = (f)$.)

Skoda's Theorem

Consider $b \subseteq \mathbb{C}[X]$. Let $d = \dim X$
Skoda's Thm, ver. I For any $m \geq d$, $\mathcal{J}(b^m) = b\,\mathcal{J}(b^{m-1})$
Hence $\mathcal{J}(b^m) = b^{m+1-d}\, \mathcal{J}(b^{d-1})$.

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(Analytically, $b = (g_1, \ldots, g_p)$
If $\int \frac{|f|^2}{\sum |g_i|^{2m}} < \infty$
then $f = \sum h_i g_i$
(w/ integrability conditions on h's))

Cor (Briançon-Skoda)
$\overline{b^m} \subseteq b^{m+1-d}$ $(\overline{b^m} \subseteq \mathcal{J}(b^m))$.

Sketch of Proof of Skoda's Theorem:
(via Lipman, Teissier)
(1) Fix generators $g_1, \ldots, g_p \in b$
and a log resolution $\mu: X' \to X$ of b
$b \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$
Write $g_i' = \mu^*(g_i) \in \Gamma(X', \mathcal{O}_{X'}(-F))$.
(2) Put the g_i' together to get

$$\underbrace{\bigoplus^{p} \mathcal{O}_{X'}}_{p \text{ copies}} \xrightarrow[(g_1', \ldots, g_p')]{\varphi} \mathcal{O}_{X'}(-F)$$

tensor with: $\otimes \mathcal{O}_{X'}(K_{X'/X} - (m-1)F)$

$$\bigoplus^{p} \mathcal{O}_{X'}(K_{X'/X} - (m-1)F) \xrightarrow{\varphi} \mathcal{O}_{X'}(K - mF)$$

Apply μ_*; get

$$\bigoplus^{p} \mathcal{J}(b^{m-1}) \xrightarrow[\text{mult by } (g_1, \ldots, g_p)]{\mu_*(\varphi)} \mathcal{J}(b^m)$$

i.e., $\operatorname{Im} \mu_*(\varphi) = b \cdot \mathcal{J}(b^{m-1}) \subseteq \mathcal{J}(b^m)$
Thm is equivalent to: $\mu_*(\varphi)$ is surjective.
(3) Resolve φ by a Koszul complex on X'. Apply vanishing to terms of complex to get surjectivity.

A variant

Given $\mathfrak{a} \subseteq \mathbb{C}[X]$, rational $c > 0$,
define "mixed" multiplier ideal $\mathcal{J}(\mathfrak{b}^m \mathfrak{a}^c)$
Take log resolution $\mu : X' \to X$

$\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-A)$

$\mathfrak{b} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-B)$

$\mathcal{J}(\mathfrak{b}^m \cdot \mathfrak{a}^c) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [mB + cA])$

Skoda's Theorem, ver. II

For $m \geq d$, $\mathcal{J}(\mathfrak{b}^m \mathfrak{a}^c) = \mathfrak{b}^{m+1-d} \mathcal{J}(\mathfrak{b}^{d-1} \mathfrak{a}^c)$

<u>Lemma</u> Fix $\mathfrak{a} \subseteq \mathbb{C}[X]$
Let $\xi' < \xi$ be consecutive jumping numbers of $\mathfrak{a}$

(*) $\mathfrak{b}^m \mathcal{J}(\mathfrak{a}^{\xi'}) \cap \mathcal{J}(\mathfrak{a}^{\xi}) \subseteq \mathfrak{b}^{m-d} \mathcal{J}(\mathfrak{a}^{\xi})$

<u>Idea</u> Can replace ξ' by $c \in [\xi', \xi)$ very close to ξ.

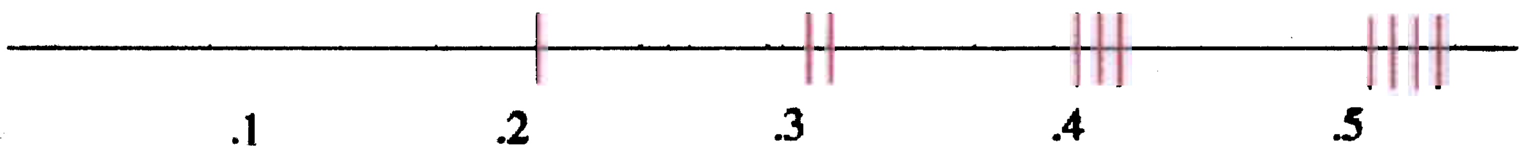
Show LHS of (*) $\subseteq \mathcal{J}(\mathfrak{b}^{m-1} \mathfrak{a}^{\xi})$ ~~spanned~~

Apply Skoda II

$\mathcal{J}(\mathfrak{b}^{m-1} \mathfrak{a}^{\xi}) \subseteq \mathfrak{b}^{m-d} \mathcal{J}(\mathfrak{b}^{d-1} \mathfrak{a}^{\xi}) \subseteq \mathfrak{b}^{m-d} \mathcal{J}(\mathfrak{a}^{\xi})$

<u>HW</u> Fix ideal ~~on~~ $\mathfrak{a}$. Starting at $d-1$, jumping numbers of $\mathfrak{a}$ are periodic with period 1, i.e., for $\xi > d-1$
ξ is jumping number for $\mathfrak{a}$ iff $\xi + 1$ is

(s^9, t^{10})



(s^3, t^{30})