Recall

G finite, k field

Greenlees Spectral Sequence

$$H^{s,t}_{\mathfrak{m}} H^*(G,k) \Rightarrow H_{-s-t}(G,k)$$

- Also versions for

$$\left.\begin{array}{l}\text{compact Lie}\\ \text{virtual duality}\\ p\text{-adic analytic}\end{array}\right\}\text{groups}$$

...

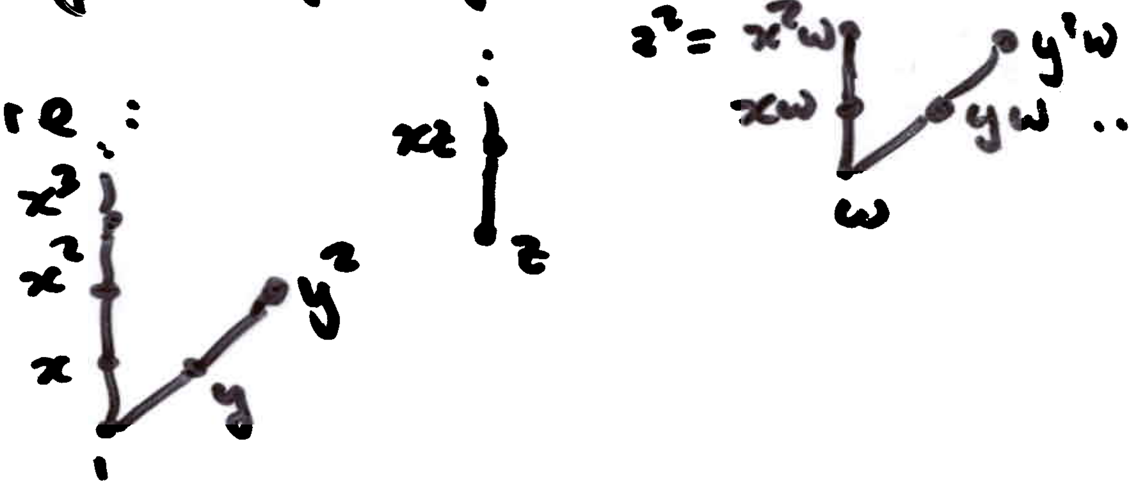
char $k = 2$ B1

Example

$$G = SD_{16} = \langle g, h \mid g^8 = 1,\ h^2 = 1,\ h^{-1}gh = g^3 \rangle$$

$$H^*(G, k) = k[\underset{①}{x}, \underset{①}{y}, \underset{③}{z}, \underset{④}{w}] \,/\, (y^3, xy, yz, z^2 + x^2 w)$$

Picture:

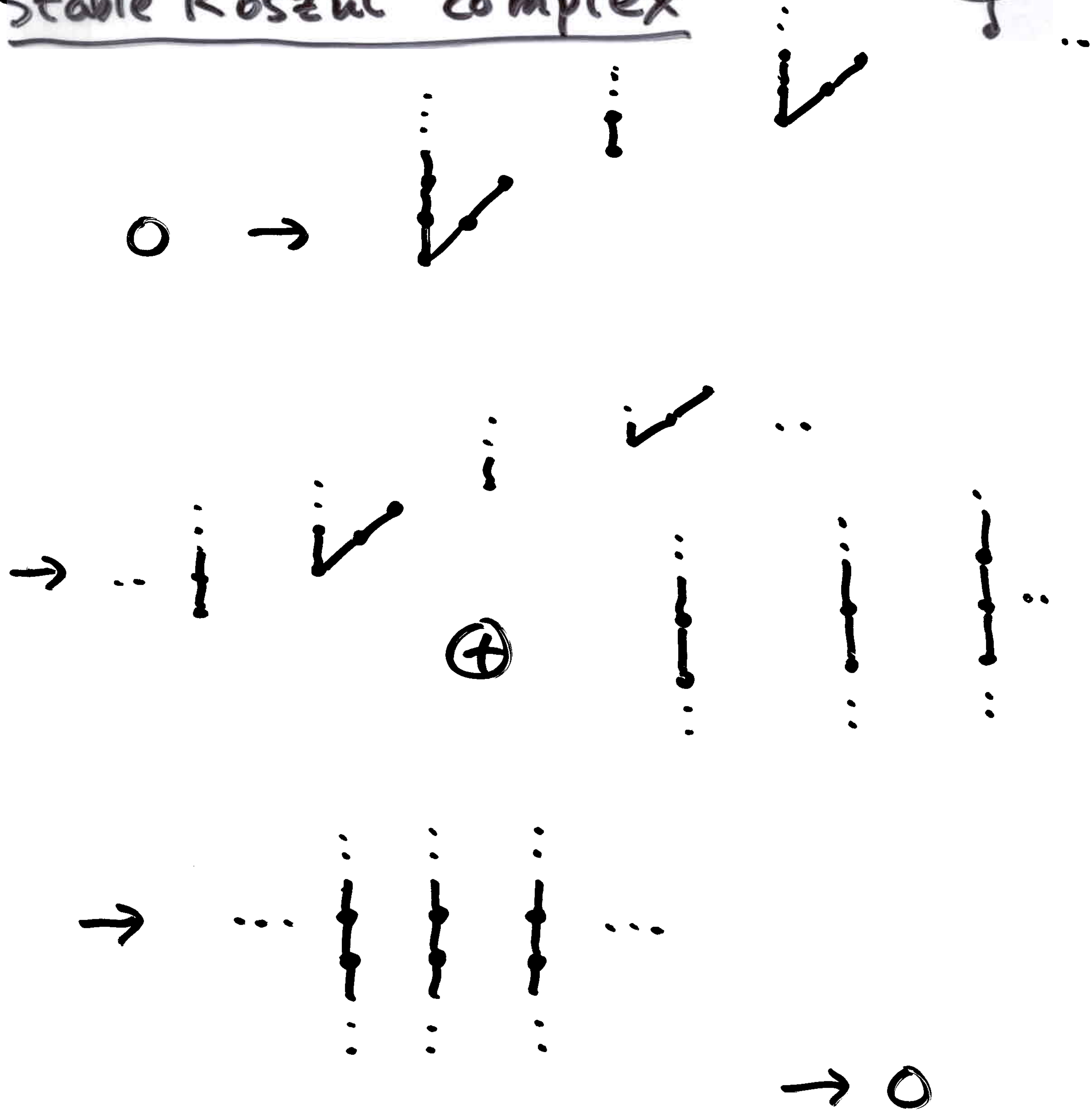


Sequence of parameters:

$w,\ x$

B2

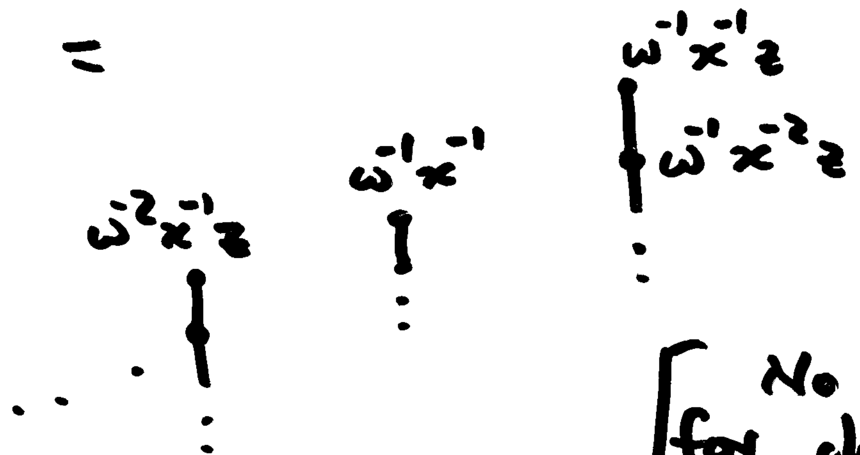
Stable Koszul complex



Local cohomology

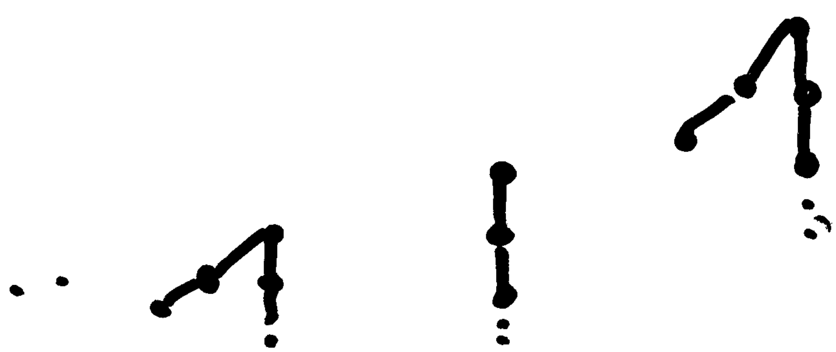
$H^1_{\mathfrak{m}} =$

$H^2_{\mathfrak{m}} =$



[No room for differentials]

Ungrading E_∞ glues $H^1_{\mathfrak{m}}$ under $H^2_{\mathfrak{m}}$ to make



Example

B4

$\Gamma_7 a_2$ order 32

H^1_m, H^2_m, H^3_m nonzero

(depth $= 1$, Krull dim $= 3$)

Nonzero differential

$d_2 : H^{1,-3}_m \to H^{3,-4}_m$

But still, everything's below

$s+t=0$ line

QUASI-REGULAR SEQUENCES

($\neq$ terminology in Matsumura's Commutative Algebra)

<u>Def^n</u> A sequence of parameters $\xi_1, \ldots, \xi_r$ is <u>filter-regular</u> if for $i = 0, \ldots, r-1$

$$(H^*/(\xi_1, \ldots, \xi_i))^j \xrightarrow{\xi_{i+1}} (H^*/(\xi_1, \ldots, \xi_i))^{j+n_{i+1}}$$

is injective for j large enough, where $n_i = |\xi_i|$.

Always exists - prime avoidance.

The sequence is <u>quasi-regular</u> if injective for $j \geqslant n_1 + \ldots + n_i$ and $(H^*/(\xi_1, \ldots, \xi_r))^j$ is zero for $j \geqslant n_1 + \ldots + n_r$.

So ξ_1 is regular, but after that, allow low degree kernel.

<u>Theorem</u> For each i, define

$$0 \to L_{\xi_i} \to \Omega^{n_i} k \xrightarrow{\hat{\xi}_i} k \to 0$$

Then $\xi_1, \ldots, \xi_r$ is quasi-regular iff for $i = 0, \ldots, r-1$ multiplication by ξ_{i+1} is injective on $H^j(G, L_{\xi_1} \otimes \cdots \otimes L_{\xi_i})$ for $j \geq n_1 + \cdots + n_i + i$.

If $\xi_1, \ldots, \xi_{r-1}$ satisfy the condition for quasi-regularity then <u>automatically</u> ξ_r does and $(H^*/(\xi_1, \ldots, \xi_r))^j = 0$ for $j \geq n_1 + \cdots + n_r$ is also automatic.

<u>Okuyama</u> & <u>Sasaki</u> have shown using a delicate transfer argument that the quasi-regularity of ξ_{r-1} is also automatic,

Since Duflot's theorem shows that we can always take g_1 to be regular, we have:

Theorem If $r_p(G) \leq 3$ then $H^*(G, k)$ has a quasi-regular sequence.

Reformulation in Local Cohomology

If M is a graded H^*-module, define

$$a^i_{\mathfrak{m}}(M) = \max\{n \in \mathbb{Z} \mid H^{i,n}_{\mathfrak{m}}(M) \neq 0\}$$

$$(\text{or } -\infty \text{ if } H^i_{\mathfrak{m}}(M) = 0)$$

<u>Theorem</u> Let G be a finite group and k a field.
The following are equivalent.

(i) $\exists$ quasi-regular sequence

(ii) Every filter-regular sequence is quasi-regular

(iii) $\forall\ i \geq 0 \quad a^i_m H^*(G,k) < 0$.

(iv) The Dickson invariants are quasi-regular

Reformulate using
<u>Grothendieck</u> <u>duality</u>:

If $R = k[\zeta_1, \ldots, \zeta_r] \subseteq H^*(G,k)$
and M is a graded H^*-module
then look at <u>minimal</u> <u>resolution</u>
of M as R-module

$$0 \to F_r \to \cdots \to F_0 \to M \to 0$$

$\beta_j^R(M)$ largest degree of a
generator of F_j (or $-\infty$ if $F_j = 0$)

$$\boxed{\begin{array}{l} \exists \text{ quasi-regular sequence} \\ \Leftrightarrow \forall i \quad \beta_i^R H^*(G,k) < \sum_{i=1}^{r} |\zeta_i| \end{array}}$$

Theorem

If $\zeta_1, \ldots, \zeta_r$ is a quasi-regular sequence in $H^*(G, k)$, then all generators have degree at most $\sum_{i=1}^{r} |\zeta_i|$ and all relations have degree at most $2\sum_{i=1}^{r} |\zeta_i| - 2$.

[Ring relations come from R-module relations + how to multiply two R-module generators.]

Castelnuovo-Mumford Regularity

Def$^{\underline{n}}$ If M is an H^*-module

$$\operatorname{Reg} M = \max_{j \geq 0} \{ a^j_{m}(M) + j \}$$

$$= \max_{j \geq 0} \{ \beta^R_j(M) - j - \sum_{i=1}^{r} (|\xi_i| - 1) \}$$

"Last survivor" of B-Carlson implies $H^{r,-r}_{m} H^*(G,k) \neq 0$

so

$$\operatorname{Reg} H^*(G,k) \geq 0$$

Conjecture $\operatorname{Reg} H^*(G,k) = 0$

Generalizations

Conjecture If G is an orientable virtual Poincaré duality group of dimension d over k then

$$\operatorname{Reg} H^*(G,k) = d.$$

Conjecture If G is a compact Lie group of dimension d and k is a field then

$$\operatorname{Reg} H^*(BG;\varepsilon) = -d.$$

Here, ε is the orientation representation of G on its Lie algebra.

Example

$$G = E_6 \qquad \dim G = 78$$

(compact, simply connected)

$$k = \mathbb{F}_2$$

Kono & Mimura (1975) calculated $H^*(BG; k)$ with one undetermined coefficient. Enough to compute that

$$\text{Krull dim} = 6$$

$$\text{depth} = 5$$

$$a_m^5 = -90 \qquad a_m^6 = -84$$

so

$$\operatorname{Reg} H^*(BG; k) = -78.$$