Bernard Teissier 1

Monomial ideals, binomial ideals, polynomial ideals

$d=1$ $\quad u^n \in k[u]$ $\qquad u^h k[[u]]$, $\quad u^n \in \{u\}$

$\sum_{i=0}^{m-1} a_i(t)u^i + u^m + \sum_{i=m+1} a_i(t) u^i$ $\qquad f = u^h I(u)$

$I(u)$ is a unit

$f = u^n$ again

$d=2$

Given two monomials is not true that one divides the other.

example: u_1^m, u_2^k

set up: $k[u_1, \dots, u_d]$

$u^m := u_1^{m_1} \cdot \ldots \cdot u_d^{m_d}$ $\qquad m = (m_1, \dots, m_d)$ $\quad m \in \mathbb{N}^d$.

we'll study monomials that divide each other.

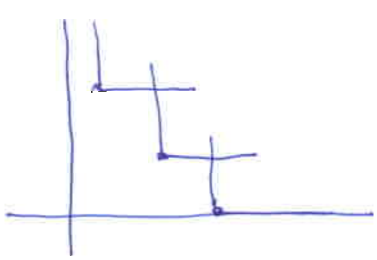
why this is usefull?

$f = \sum f_p u^p \in k[[u]]$ $\qquad \underline{u} = (u_1, \dots, u_d)$

$f_p \in k$

$u_1^{m_1}, \dots, u_d^{m_d}$ generate all the monomial that appear in f

Picture = staircase



$\mathrm{Supp} f = \{p \mid f_p \neq 0\} \subset \mathbb{N}^d$

each $p \in \mathrm{supp} f$ takes

$P_+ \mathbb{R}_+^d$

$u^{m_1}, \dots, u^{m_d}$

Newton's algorithm:

$$f(x,y)=0 \qquad f\in k[[x,y]]$$

$$\exists\, y(x)? \quad \text{s.t. } f(x,y(x))\equiv 0$$

this problem doesn't have always a solution.

Newton was looking for solution as:

$$y=\varphi(x^{\frac{1}{m}}) \quad \text{for some } m.$$

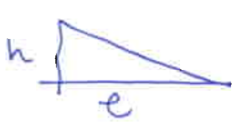
$$f=\sum a_{ij}x^iy^j \qquad y=x^{\nu}(c_0+\varphi_1(x^{\frac{1}{m}})) \quad \nu\in\mathbb{Q}_+$$

$$f=\sum a_{ij}x^{i+\nu j}(c_0+\varphi_1(x^{\frac{1}{m}}))^j$$

$$\mu=\min_{(i,j)\text{ s.t. } a_{ij}=0}(i+\nu j)$$

$$f=x^{\mu}\sum_{i+\nu j=\mu}a_{ij}c_0^j+x^{\mu}h(x^{\frac{1}{m}})$$

ν has to be the inclination of a side of the $\underbrace{\text{hull - staircase}}_{\text{Newton polygon}}$



$$\frac{h}{e}=\nu$$

change the coordinate:

$$\begin{cases} x=x_1^{e} \\ y=x_1^{h}(c_0+y_1) \end{cases}$$

$$f\longrightarrow f_1$$

either k drops or k doesn't drop and ν increases strictly.

$\varphi(x,y) = f_1(x,y) \dots f_t(x,y)$

where f_i are irreducible curves.

for each $f_k(x,y)$ $\quad \begin{cases} x = t^{m_k} \\ y = \varphi(t) \end{cases} \longleftrightarrow y = \varphi(x^{1/m_k})$

for irreducible curves the Newton polytope

has one finite side $\nu = \cancel{\frac{m_t\varphi}{m/t}} \quad \frac{m_t \varphi}{m}$.

The converse is not true.

Let $u_1 = y_1^{a_1^1} \dots y_d^{a_1^d}$

$\vdots$

$u_d = y_1^{a_d^1} \dots y_d^{a_d^d}$

$\mathbb{A}^d \longrightarrow \mathbb{A}^d$

$\underline{y} \longrightarrow \underline{u}$

to be birational iff $\det(a^1, \dots, a^d) = \pm 1$

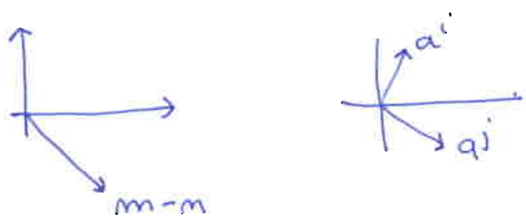
need also surjectivity.

$u^m, u^h \qquad m \in \mathbb{N}^d \quad h \in \mathbb{N}^d$

$u^m \longrightarrow y_1^{\langle a^1, m\rangle} \dots y_d^{\langle a^d, m\rangle} = (u^m)'$, where $\langle a, m\rangle = \sum_i a^i m_i$.

$(u^h)' \mid (u^m)' \iff \forall i \quad \langle a^i, n\rangle < \langle a^i, m\rangle$

$\iff \forall i \quad \langle a^i, m-n\rangle \geq 0$



take the hyperplane in ~~of~~ the weight space dual to $m-n$ (H_{m-n})

the condition says that all a^i should be on the same side of H_{m-n}.

Fan: in $\mathbb{R}_+^d$

$\Sigma_1 = (\sigma_\alpha)$ each $\sigma_\alpha = \langle a^{1,\alpha}, \dots, a^{i,\alpha} \rangle$

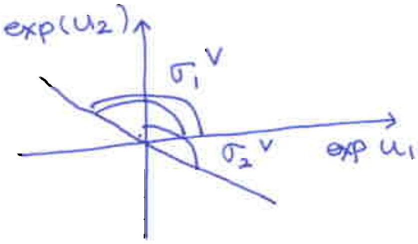
1) $a^1, \dots, a^i$ are part of a basis of the integral lattice

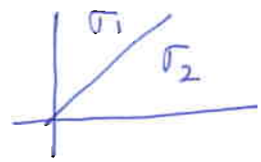
2) $\bigcup \sigma_\alpha = \mathbb{R}_+^d$

3) $\sigma_\alpha \cap \sigma_\beta \neq (0) \Rightarrow \sigma_\alpha \cap \sigma_\beta$ is a face

$\sigma = \langle a^1, \dots, a^d \rangle$ $\qquad \sigma^\vee = \{ m \mid \langle m, v \rangle \geq 0 \ \forall v \in \sigma \}$

example:





properties:

- $\sigma \subset \sigma' \Rightarrow \sigma^\vee \supset (\sigma')^\vee$

$\rightarrow k[\sigma^\vee \cap \mathbb{Z}^d] = k[y_1, \dots, y_d]$

In the above example: $k[y_1, y_2] = k[u_1, u_1^{-1}u_2] \subset k[u_1, u_1^{-1}u_2, (u_1^{-1}u_2)^{-1}]$
$k[y_1', y_2'] = k[u_2, u_2^{-1}u_1] \subset$

In general:

if $|\Sigma_1| = \mathbb{R}_+^d$

then we have a birational map: $Z(\Sigma_1) \to A^d$

to get $\bigcup u^m \mathcal{O}_{\Sigma(\Sigma_1)} \mid u^m \mathcal{O}_{\Sigma(\Sigma)}$ or the opposite

$\Longleftrightarrow$ any σ in the fan has lies entirely

on one side of $H_m - m$

any number of monomial

$u^{m_1} \dots u^{m_e} \longrightarrow$ you need to check all difference $m_s - m_t$ $s \neq t$.

a Ring V in which given any ~~[illegible]~~ two elements one divides the other is a valuation ring.

The ring V has to be a commutative domain.

Suppose u^p is another monomial.

Suppose that $\langle a^i, p \rangle \geq \min_s \langle a^i, m_s \rangle$, $\forall i$

$\forall$ comes in the fan.

$\Rightarrow$ u^p is going to be in the ideal.

Prop: $u^{m_1}, \dots, u^{m_k}$ a set of monomial

$\Sigma_1 \subset \mathbb{R}_+^d$ is a fan compatible with the hyperplan $H_{m_j - m_t}$ $\forall s, t$ $s \neq t$

(a) $a\,\mathcal{O}_{Z(\Sigma)}$ is locally principal

(b) $\pi(\Sigma)_* \left(a\,\mathcal{O}_{\Sigma(\Sigma)} \right)$ = monomial ideal generated by all monomials in the convex hull of $\bigcup (m_i + \mathbb{R}_+^d)$

$\overset{\shortparallel}{\bar{a}}$

integral closure of a.