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Let V be a finite-dimensional vector space over k
Let $G \to V$ be a finite group acting on V.
Assume always that char $k \nmid |G|$
Maschke's Theorem: $V = \oplus V_i$ ← each invariant wrt G
where the V_i's are irreducible

Let $k[V]$ be the algebra of polynomial functions,
$k[V]^G$ the ring of invariants
can think of $k[V] \leftarrow\!\!\!-\!\!\!\rightarrow V$ as functions on V
then $k[V]^G \leftarrow\!\!\!-\!\!\!\rightarrow V/G$ are functions on orbits
let $I_G = (k[V]^G_+) \cdot k[V]$
the ideal in $k[V]$ generated by
the homogeneous maximal ideal of $k[V]^G$
(i.e., generated by homogeneous invariants w deg>0)
then $R_G = k[V]/I_G$ is the ring of coinvariants

Lemma (Hilbert): $f_1, \dots, f_r$ homogeneous $\in k[V]^G_+$:
$k[V]^G = k[f_1, \dots, f_r] \Leftrightarrow I_G = (f_1, \dots, f_r)$
Proof $\Rightarrow$: clear
$\Leftarrow$: if not, let d = smallest degree of
$h \in k[V]^G \setminus k[f_1, \dots, f_r]$
h is not constant, so it is in I_G
$h = \sum a_i f_i$, a_i homogeneous of deg $= d - \deg f_i$
apply Reynolds operator $Rf = \frac{1}{|G|} \sum_{g \in G} g \cdot f$
which projects $k[V] \to k[V]^G$
since h, f_i are invariant, $h = \sum_i R(a_i) f_i$
since $\deg a_i < d$, and $R(a_i)$ is invariant,
$R(a_i) \in k[f_1, \dots, f_r]$, so $h \in k[f_1, \dots, f_r]$. ∎

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Noether's Thm $k[V]^G$ is generated by elts of degree $\leq |G|$.
(classical proof is in char 0)
more recent proof using Derksen's Lemma - any char
look at coordinate ring on $k[V\times V]$
Coordinates $\underline{x}$ on $V\times 0$
$\underline{y}$ on $0\times V$

$V_g = \{(\underline{x}, g\underline{x})\} \overset{?}{=} V(\underbrace{y_i - gx_i : i=1,\dots,\dim V}_{J_g})$

the Derksen arrangement $\bigcup_{g\in G} V_g$ ~~has ideal~~ is the variety of
$J = \bigcap J_g \subseteq k[V\times V] = k[V\times V]^{G} = k[\underline{x},\underline{y}]$

Lemma (Derksen) $(J+(\underline{y}))\cap k[\underline{x}] = I_G$
Proof ($\supseteq$) $f\in k[V]^G_+$ $\quad f(\underline{x}) - f(\underline{y}) \in J$
($\subseteq$) $f(\underline{x}) = \sum a_i(\underline{x}) \underbrace{b_i(\underline{y})}_{\in (\underline{y})} + p(\underline{x},\underline{y})$

apply R_y, the Reynolds operator in only the y_i's
$f(\underline{x}) = \sum a_i(x) \underbrace{R_y b_i}_{\in I_G(\underline{y})} + \underbrace{R_y p(\underline{x},\underline{y})}_{\in J}$

$\xrightarrow[y\mapsto x]{} f(\underline{x}) = \sum a_i(\underline{x}) R b_i(\underline{x})$

Lemma Let $J' = \prod_g J_g \quad (J'+(y))\cap k[\underline{x}] = (\underline{x})^{|G|}$
Proof $(J'+(y))\cap k[\underline{x}] = \{f(\underline{x},0) : f(\underline{x},\underline{y})\in J'\}$
($\subseteq$) easy
($\supseteq$) Let $\underline{x}^\alpha$ be a monomial of degree $|G| \overset{def}{=} d$
write $\underline{x}^\alpha = x_{i_1} x_{i_2}\cdots x_{i_d}$, $\quad G = \{g_1,\dots,g_d\}$
$f(x,y) = \prod_{j=1}^{d} (x_{i_j} - g_j y_{i_j}) \in J'$, $\quad f(\underline{x},0) = \underline{x}^\alpha$

$J' \subseteq J \Rightarrow (\underline{x})^d \subseteq I_G$, R_G is 0 in degrees $> |G|$. ∎

now let $k = \mathbb{C}$, $V = E = \{(P_1,\dots,P_n)\}$, $P_i \in \mathbb{C}^2$
$G = S_n \quad \mathbb{C}[E] = \mathbb{C}[\underline{x},\underline{y}]$, $\sigma x_i = x_{\sigma(i)}$, $\sigma y_i = y_{\sigma(i)}$

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Now look at case of points on a line again:
$P_i \in \mathbb{C}^1$: $\mathbb{C}[x_1, \ldots, x_n]$
$\mathbb{C}[\underline{x}]^{S_n} = \mathbb{C}[e_1, \ldots, e_n]$,
where $e_k = e_k(\underline{x})$ is the kth elementary symmetric function

$I_{S_n} = (e_1, \ldots, e_n)$
R_{S_n} is a C.I. - complete intersection
Hilbert series of $R_{S_n} = \dfrac{(1-q)\cdots(1-q^n)}{(1-q)^n}$

$= [n]_q! = [n]_q[n-1]_q \cdots [1]_q$
$[k]_q = (q^{k-1} + q^{k-2} + \cdots + 1)$
In fact as an S_n-rep, $R_{S_n} \cong \mathbb{C}S_n$
Furthermore,

$$\sum (\mathrm{mult}((R_{S_n})_m, V^\lambda) q^m) = f_\lambda(q)$$

partition indexing irrep

$$= S_\lambda(1, q, q^2, q^3, \ldots) \cdot (1-q) \cdot \cdots \cdot (1-q^n)$$

similar facts hold for any G generated by reflections

the transposition (i,j) $x_i \longleftrightarrow x_j$ fixes (pointwise) every point on the hyperplane $x_i = x_j$
- an (order 2) linear automorphism fixing a hyperplane is a reflection
cf. theory due to Shepard-Todd, Chevalley, Steinberg

Case $V = E = \mathbb{C}^{2n}$ $\mathbb{C}[\underline{x}, \underline{y}]$ points on a plane

$\mathbb{C}[\underline{x}, \underline{y}]^{S_n} = \mathbb{C}[p_{r,s} : 1 \leq r+s \leq n]$,
$p_{r,s} = \sum_i x_i^r y_i^s$ · polarized power sum
minimal generators (cf. Weyl, eg Classical Groups)
also true of $\mathbb{C}[\underline{x}, \underline{y}, \ldots, \underline{z}]$
Not much else is known. Syzygies? Free resolution?

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Remark Hochster-Roberts:
$\mathbb{C}[\underline{x},\underline{y}]$ is Cohen-Macaulay, Gorenstein
here $S_n \subseteq SL(E)$
because a transposition $i \leftrightarrow j$ interchanges both
$x_i \leftrightarrow x_j$, $y_i \leftrightarrow y_j$
- hence has determinant 1
Hilbert series $= h_n(1, q, q^2, \dots, t, qt, q^2t, \dots, t^2, qt^2, \dots)$
complete homogeneous symmetric function
$R_{S_n} = \mathbb{C}[\underline{x}, \underline{y}]/I_{S_n}$ $\quad I_{S_n} = (p_{r,s} : 1 \le r+s \le n)$
Thm $\dim_{\mathbb{C}} R_{S_n} = (n+1)^{n-1}$ $\quad$ regularity $\binom{n}{2}$
$\dim_{\mathbb{C}} R^{\varepsilon}_{S_n} = C_n = \frac{1}{n+1}\binom{2n}{n}$, nth Catalan #
alternating part
$R_{S_n} \underset{S_n}{\cong} \varepsilon \otimes \mathbb{C} \cdot (Q/(n+1)Q)$ where $Q = \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$
as an S_n-module
root lattice of type A_{n-1}
for which S_n is the Weyl group
# of inversions of F
$\sum \dim_{\mathbb{C}} (R_{S_n})_{r,-} q^r = \sum_{F \text{ a forest}} q^{\mathrm{inv}(F)}$

Problems What about other Coxeter groups?

cf. recent preprint by Iain Gordon - a weak yes
- analogous results for all Coxeter groups

Conjecture: Derksen's arrangement: $W \subseteq E \times E$
$\mathbb{C}[W] = \mathbb{C}[\underline{x}, \underline{y}, \underline{x}', \underline{y}']/?$ is a free $\mathbb{C}[\underline{x}]$-module