Teissier 2.

Monomial ideals, binomial ideals,

Remorse:

$$u^{m_1},\dots,u^{m_d^{h}} \in k[u_1,\dots,u_d]$$

$$m_s - m_t \longrightarrow H_{m^s-m^t} \subset \text{Weight space} : (\mathbb{R}^d)^\vee \supseteq \mathbb{Z}^d$$

$$\mathbb{R}^{\vee d} \supset \underline{\cup H}$$

if σ is not regular then $k[\sigma^\vee \cap M]$ is not a polynomial ring

$$Z(\Sigma_0) \longrightarrow \mathbb{A}^d$$

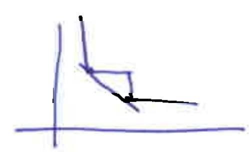
$\nearrow$

$Z(\Sigma)$ find a fan Σ such that $Z(\Sigma) \to Z(\Sigma_0)$ is a resolution of singularity.

if you have a fan not regular, we can still prove Briançon Skoda THM for monomial ideals:

$$u^p \in \overline{(u^{m_1},\dots,u^{m_h})}$$

$$p \in \operatorname{conv}^+(\cup(m_i + \mathbb{R}_+^d))$$



Caratheodory's THM

$E \subset \mathbb{R}^d$ $\forall p \in \operatorname{conv}(E)$ $\exists x_0,\dots,x_d \in E$ s.t.

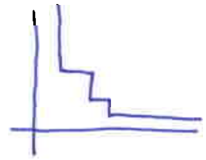
$p \in \operatorname{Conv}(x_0,\dots,x_d)$

Improvement

Suppose $\exists p \in \operatorname{conv} E$ for which you cannot find d points in E containing p in their convex hull

$\Rightarrow$ E has at least $d+1$ connected components using the improvement:

$$u^p \in \overline{(u^{m_1}, \dots, u^{m_n})}$$



$$p \in \text{convex hull}(m_1, \dots m_n)$$

<u>Claim</u> "the above of the staircase" is connected

by previous improvement we can find d generators

$$p = \sum_{i=1}^{d} x_i m_i \qquad \sum_{i=1}^{d} x_i = 1, \ \exists\, x_i \geq \frac{1}{d}, \text{ suppose } x_1 \geq \frac{1}{d}$$

$$(u^p)^d \in (u^{m_1}, \dots, u^{m_n})$$

$$dp = m_1 + \dots +$$

$$dp \in E \Rightarrow \quad \overline{(a^d)} \subseteq a$$

The proof work for rational singularity.

Going back to yesterday:

$$u^m - cu^n \longmapsto Y_1^{\langle a^1, m\rangle} \cdots Y_d^{\langle a^d, m\rangle} \underbrace{\left(Y_1^{\langle a^1, m-n\rangle} \cdots Y_d^{\langle a^d, m-n\rangle} - c\right)}$$

$$f = \sum f_p u^p$$

assume the cone $\langle a^1, \dots, a^d\rangle$ is one side of H_{m-n}

This part is not singular

$$Y^{\langle a^?, m\rangle} \mid Y^{\langle a, m\rangle}$$

$$f = \sum f_p u^p \rightsquigarrow a = (u^{m_1}, \dots, u^{m_n})$$
$$f_p \in k$$

$$f \circ \pi(\Sigma_1)\Big|_{\sigma = \langle a^1, \dots, a^d\rangle} = \sum f_p Y_1^{\langle a^1, p\rangle} \cdots Y_d^{\langle a^d, p\rangle}$$

$$i \longrightarrow \min_{p \text{ s.t. } a_p \neq 0} \langle a^i, p \rangle =: H(a^i)$$

$$H(e) = \min_{p \text{ s.t. } a_p \neq 0} (e(p))$$

$$f|_{Z(\sigma)} = y_1^{H(a^1)} \; y_d^{H(a^d)} \Big(\underbrace{\Sigma_1 f_p y_1^{\langle a^1, p\rangle - H(a^1)} \cdots y_d^{\langle a^d, p\rangle - H(a^d)}}_{\tilde{f}} \Big)$$

in $Z(\sigma)$ $y_1, \dots, y_d$

$\exists$ a stratification, Pick $I \subset \{1, \dots, d\}$

$S_I = \{$p.t when $y_i = 0$ $i \in I$, $y_i \neq 0$ $i \notin I\}$

$(S_I)_{I \subset \{1, \dots, d\}}$

There is a correspondence between the S_I's and the compact faces of the Newton polyhedra of f.

Define indeed an equivalent relation on $\check{\mathbb{R}}^d_+$ saying that:

$$e \sim e' \iff \{p \in N(f) / e(p) = H(e)\} = \{p \in N(f) / e'(p) = H(e')\}$$

Fact: Every compact ~~space~~ face γ of NP (Newton polyhedra) is obtained as $\{p$ s.t. $e(p) = H(e)$ for ~~some~~ all e in some cone $\mathcal{T}_i \subseteq \mathbb{R}^d\}$

Since the fan is compatible $\sigma \in \Sigma$

1. $$\tilde{f}|_{S_I} = \sum_{\substack{(j_1, \dots, j_e) \\ \cap \{1, \dots, d\} \setminus I \\ p \in \gamma_I}} f_p \, y_{j_1}^{\langle a^{j_1}, p\rangle - H(a^{j_1})} \cdots y_{j_e}^{\langle a^{j_e}, p\rangle - H(a^{j_e})}$$

2. $$\tilde{f}|_{S_I} = \left(\begin{matrix}\text{sto.} \\ \text{Transform} \\ \text{of}\end{matrix} \sum_{p \in \gamma_I} f_p u^p \right) \Bigg|_{\substack{\text{restricted} \\ \text{to } S_I}} = f_{\gamma_I}$$

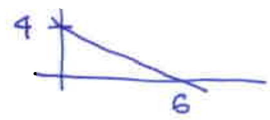
1., 2. } THM:

if f_{γ_I} has no singularities on $(k^*)^d$ then

$\tilde{f}|_{S_I}$ has no singularities.

Example:

$$(y^2 - x^3)^2 - x^5 y = 0$$



there is just one compact face.

Everything depends on the choose of coordinates.

THM. For every compact face γ of $NP(f)$, f_γ non singular on $(k^*)^d$, the map $Z(\Sigma) \xrightarrow{\pi(\Sigma)} \mathbb{A}^d$, for Σ regular face compact with $NP(f)$ ($H(e)$ linear in each $\sigma \in \Sigma$) is an embedded resolution of the hyp $f = 0$

$\pi(\Sigma)$ is a non singular map of Non singular variaties iff the strict transformation of $f = 0$ is non singular and $\pitchfork$ exce. divisor.

Suppose $f_1, \dots, f_c$ are functions

each newton polyhedra has ~~has~~ its compact faces.

the Jacobian matrix is:

$$\begin{matrix} \frac{\partial f_1|_{\gamma_1}}{\partial u_1} & \dots & \frac{\partial f_c}{\partial u_1}\Big|_{\gamma_c} \\ \\ \frac{\partial f_1}{\partial u_d}\Big|_{\gamma_1} & \dots & \frac{\partial f_c}{\partial u_d}\Big|_{\gamma_c} \end{matrix}$$

if $\forall$ any choice of $\gamma_1 \cdots \gamma_c$ the Jac matrix

has $(d-h)$-minor non vanishing at $(k^*)^d$

Then · embedded resolution.

<u>Problem</u>: any algebraic variety X (reduced and equidimensional)

$x \in X$, $\exists$ an étale nbd U of x in X in such a way

$$\begin{tikzcd} & \mathbb{A}^N \\ x \in U \arrow[ur] \arrow[r] & \mathbb{A}^d \arrow[u] \end{tikzcd}$$

that $\exists$ a coordinates in $\mathbb{A}^N$

such that X is non degenerate

at x in $\mathbb{A}^N$.

Maybe yes.

In all the talks k is algebraically closed