Teissier 3

Monomial ideals, binomial ideals, polynomial ideals, etc

REMORSE:

$$\begin{tikzcd} Z(\Sigma) \arrow[r, "\pi(\Sigma)"] & \mathbb{A}^d \\ X' \arrow[u, hook] \arrow[r, "\pi(\Sigma)"] & X \arrow[u, hook] \end{tikzcd}$$

Is π an isomorphism outside Sing X?

THM (González Pérez) X is irreducible binomial variety, $X \subseteq \mathbb{A}^d(k)$; $k = \bar{k}$

X is non degenerate and π is an isomorphism outside Sing X.

$(u^{m^s} - \lambda_{mn} u^{n^s})$ prime ideal. $1 \leq s \leq L$

This ideal satisfies the condition of THM in Talk 2.

How about dim of the ideal?

Eis.-Stur.

$$\mathcal{L} = \langle m^s - n^s \rangle \subseteq \mathbb{Z}^d$$

X is irreducible $\iff \mathcal{L}$ is a direct factor in $\mathbb{Z}^d$

$\operatorname{rk} \mathcal{L} = \operatorname{codim} X$

Let $c = \operatorname{rk} \mathcal{L}$. $L' \subset L$ $|L'| = c$

$$u_{i_1} \cdots u_{i_c} J_{I,L'} = \prod_{e \in L'} u^{m^e} \operatorname{Det}_{I,L'}(\langle m^s - n^s \rangle)$$

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Let $u^m - \lambda_{mn} u^n + v u^p$, suppose that $\langle a^i, p \rangle \geq \langle a^i, m \rangle$ for all i. This means that, after the transformation in the y's, we get:

$$y^{\bullet}(y^{\bullet} - \lambda_{mn} + v y^{\bullet})$$

The goal is to do some kind of deformation into binomial ideals. and then the map the resolves the binomial variety works in general.

Consider:

1 dimensional complete noetherian integral domain, algebraically closed residive field, equal characteristic.

$$k[[u_1,u_2]]/f(u_1,u_2) = R$$

$R \subseteq k[[t]] :=$ integral closure of R in its quotient field (finite R-module).

$\exists$ a conductor element, i.e. $\exists$ $h \in R$ s.t. $h\,k[[t]] \subset R$

$\Gamma :=$ values of t-adic valuation on $R \subseteq \mathbb{N}$.

$\Gamma = \langle \gamma_1, \dots, \gamma_{g+1} \rangle$ where γ_{i+1} is not in the subgroup generated by $\langle \gamma_1, \dots, \gamma_i \rangle$.

but
$$\begin{cases} m_i \gamma_i = \sum_{k=1}^{i-1} e_k^{(i)} \gamma_k & i = 2, \dots, g \\ \text{other relations.} \end{cases}$$

Consider the curve $C^\Gamma := \{ u_1 = t^{\gamma_1}, \dots, u_{g+1} = t^{\gamma_{g+1}} \}$

this curve C^Γ is defined by a finite number of binomial relations. It is also true that C^Γ is irreducible curve.

Therefore we can build a fan and get:

$$\begin{tikzcd} Z(\Sigma) \arrow[r] & \mathbb{A}^{g+1} \\ W^\Gamma \arrow[u] \arrow[r] & C^\Gamma \end{tikzcd}$$

What is the relation between the two curves?

By definition:

For each i, $1 \le i \le g+1$ there exists a $\xi_i(t) \in R$

$$\xi_i = t^{\gamma_i} + \sum_{j>\gamma_i} c_j^{(i)} t^j$$

divide by: $\xi(vt)\, v^{-\gamma_i}$, get

$$\xi(vt)\, v^{-\gamma_i} = t^{\gamma_i} + \sum c_j^{(i)} v^{j-\gamma_i} t^j$$

this is a parametrization from the curve to the monomial curve ($v = 0$), while $v \neq 0$ gives a curve isomorphic to the original). I want a system of equation as: $u^{m^s} - u^{m^s} - \sum c_p^{(s)}(v) u^p$

A different approach:

$R \subset V$ ~~...~~ where V is a valuation ring.

In V: $a, b \in V$ $a \le b$ iff $a | b$

$a \sim b$ iff $a|b$ and $b|a$

~~...~~, $V \setminus \{0\}/\sim \; = \phi_+ \cup \{0\}$ positive part of a totally ordered group ϕ

$$K^* \xrightarrow{\nu} \phi$$

$\parallel$

multiplicative group of K

$K = \text{tot}\, V$.

this map has the following property:

$\nu(x) = x$ modulo $\sim$

$\nu(xy) = \nu(x) + \nu(y)$

$\nu(x+y) \ge \min(\nu(x), \nu(y))$

$P_\varphi(R) = \{x \in R \text{ s.t. } \nu(x) \ge \varphi\}$

$P_\varphi^+(R) = \{x \in R \text{ s.t. } \nu(x) > \varphi\}$

$$\bigoplus_{\varphi \in \Phi_+ \cup \{0\}} \mathcal{P}_\varphi(R) \big/ \mathcal{P}^+_\varphi(R) \cong k[u] \big/ \langle u^m - \lambda_{mm} u^n \rangle$$

Compute $gr_{(t)} R = \mathbb{C}[t^{\gamma_1}, \dots, t^{\gamma_{g+1}}] \subset \mathbb{C}[t] = gr_{(t)} k[[t]]$.

If $R \supset \sigma_m$, $\cap \sigma_m = 0$

$$\mathcal{R}_\sigma(R) = \bigoplus_{m \in \mathbb{Z}} \sigma_m v^{-m} \subset R[v, v^{-1}]$$, say $\sigma_m = R$ when $m \leq 0$

In the special case when $\sigma_m = I^m$ then

$$\mathcal{R}_I(R) = R[v, I v^{-1}] \subset R[v, v^{-1}]$$

Since R is equicharacteristic, $k \subset R \Longrightarrow$

$k[v] \hookrightarrow \mathcal{R}_\sigma(R)$

So there is a map $\operatorname{Spec}(\mathcal{R}_\sigma(R)) \longrightarrow \operatorname{Spec}(k[v])$
flat (faithfully)

There is also a map:

$$\mathcal{R}_\sigma(R) \longrightarrow gr_\sigma(R)$$

$$x v^{-m} \longrightarrow in_\sigma(x) \in \sigma_m / \sigma_{m+1}$$

$x v^{-m} = (x v) v^{-(m+1)}$

So the kernel is generated by v

moreover:

$$x_m v^{-m} \longrightarrow x_m c^{-m} \in R$$

The kernel is $(v - c)\mathcal{R}_\sigma(R)$.

Map $k[[u_1, \dots, u_{g+1}]] \longrightarrow R$

$u_i \longrightarrow \xi_i$

Get a valuation in R and a filtration on R:

$$\mathrm{gr}_{\nu}\, k[[u]] = k[u_i] \longrightarrow \mathrm{gr}_\nu R$$

$$((u^{m} - u^{n})) \subset k[u_i] \longrightarrow \mathrm{gr}_\nu R$$

Since we were looking for a system of equations of the type:

$$u^{m^s} - u^{n^s} + \sum c_p^{(s)}(v) u^p = 0$$

we need weight $(u^p) >$ weight $(u^{m^s}) = w(u^{m^s})$.

Other said, we want:

$$\langle p, a^i \rangle \geq \langle m^s, a^i \rangle \quad \forall i.$$

Example: The simplest degenerate plane curve:

$$(v_1^2 - v_0^3)^2 - v_0^5 v_1 = 0$$

$\Gamma = \langle 4, 6, 13 \rangle$.

The graded ring

$$u_1^2 - u_0^3 = 0$$

$$u_2^3 - u_0^5 u_1 = 0$$

parameter v:

$$u_1^2 - u_0^3 - \underbrace{v u_2}_{\text{small perturbation}} = 0$$

$$u_2^3 - u_0^5 u_1 = 0 \quad ..$$