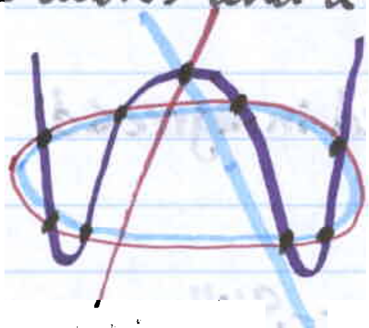
Ruchira Datta MSRI Comm. Alg. Workshop Notes

Everything is homogeneous:

A case of 9 points in $\mathbb{P}^2$: ideal generated by 2 cubics and a quartic



resolution:

$$0 \to \begin{matrix} S(-4) \\ \oplus \\ \underbrace{S(-6)} \end{matrix} \to \begin{matrix} S(-3)^2 \\ \oplus \\ \underbrace{S(-4)} \end{matrix} \to S \to S_X \to 0$$

Betti diagram:

	S		
0	1	–	–
1	–	–	–
2	–	2	1
3	–	1	–
4	–	–	1

How long is the resolution?

Auslander-Buchsbaum:

proj dim $M = \underbrace{\dim S}_{r+1} - \operatorname{depth} M$

How high is the Betti diagram? – reg M, the regularity of M

Suppose $S = K[x_0, \ldots, x_r]$

M an ideal whose generators are of degree $\le d$

1) Suppose $M \subset S$ is primary to $(x_0, \ldots, x_r) = \mathfrak{m}$

Then reg M = the smallest t s.t. $M \supset \mathfrak{m}^t$

Ruchira Datta Comm. Alg. Workshop Notes

Corollary In this case:
if $M \supset$ a regular sequence of degrees $a_1, \dots, a_r$
then $\operatorname{reg} M \leq 1 + \sum (a_i - 1)$

In general: $M \subset S$, generated in degree $\leq d$.
Then $\operatorname{reg} M \leq (2d)^{2^{r-1}}$

Example Mayr-Meyer, Bayer-Stillman
$\exists M$ with $d=4$, $\operatorname{reg} M \geq 2^{2^{(r+1)/10}} + 1$;
$n = (r+1)/10$, $10n-6$ generators

Conjecture (maybe Lazarsfeld's?)
If M is a prime ideal, ($K = \bar{K}$), then
if M contains a regular sequence of
length = codim M, degrees a_i, then
$\operatorname{reg} M \leq 1 + \sum (a_i - 1)$

Theorem Bertram, Ein, Lazarsfeld:
True in char 0 if $V(M) \subset \mathbb{P}^r$ is smooth

Theorem (Castelnuovo: smooth curves in $\mathbb{P}^3$,
Gruson-Lazarsfeld-Peskine: general)
$K = \bar{K}$. If P is a 2-dimensional prime ideal
then $\operatorname{reg} P \leq \deg P - r + 2$.
The twisted cubic in $\mathbb{P}^3$ shows the bound is tight.

Ruchira Datta MSRI Comm. Alg. Workshop Notes

Conjecture (Eisenbud, Goto)
P prime $\Rightarrow$ reg $P \leq$ deg P − codim $P + 1$

<u>Thm</u> (Derksen & Sidman)
Let M be the ideal of a union of planes in $\mathbb{P}^r$
Then reg $M \leq$ # of planes

Thm of Mumford: char 0, smooth
shows reg $P = O(\deg P)$

Peeva-Sturmfels: conjecture true for
codim 2 toric varieties

If $0 \to F_t \to \cdots \to F_0 \to M \to 0$ is a minimal free resol'n
with $F_i = \bigoplus_j S(-a_{ij})$
then reg $M = \max\{a_{ij} - i\}$

E.g. in our example, the regularity is 5

See Hartshorne problem list - J. of Topology - 1980
Horrocks / Buchsbaum - Eisenbud conjecture
about rank of syzygies

regularity is best complexity measure
for processes in computational commutative algebra
e.g. Gröbner bases

Ruchira Datta MSRI Comm. Alg. Workshop Notes

Linear Strand of a Resolution

Let M be a f.g. (finitely generated) graded module over $S = K[x_0, \ldots, x_r]$

assume $M = M_0 \oplus M_1 \oplus \cdots$ with $M_0 \neq 0$
(wlog - could just shift)

can't have anything of deg ≤ 0 here

$$\begin{array}{ccccccccc}
\cdots \to & S^{\beta_{22}}(-2) & \to & S^{\beta_{11}}(-1) & \to & S^{\beta_{00}} & \to & M & \to 0 \\
& \oplus & & \oplus & & \oplus & & & \\
& \vdots & & \vdots & & S(-1)^{\beta_{01}} & & & \\
& & & & & \oplus & & & \\
& & & & & \vdots & & &
\end{array}$$

The linear strand $\cdots \to S^{\beta_{22}}(-2) \to S^{\beta_{11}}(-1) \to S^{\beta_{00}}$ is a subcomplex.

A linear complex is one of the form
$$(\cdots \to S(-1)^{a_1} \to S^{a_0})(d)$$
(the matrices here are matrices of linear forms)

Prop Every min free resol'n has a filtration by linear complexes.
(i.e., the successive quotients are linear complexes)

In the example, the point  is distinguished.

Ruchira Datta MSRI Comm. Alg. Workshop Notes

Example bis Resolution of $I(3)$ (9 points)

ideal of the distinguished point

$$\begin{tikzcd} S(-1) \arrow[r, "\binom{l_1}{l_2}"] & S^2 \arrow[r] & I(3) \\ \oplus & \oplus & \\ S(-3) \arrow[r] \arrow[ur] & S(-1) & \end{tikzcd}$$

the linear strand has picked it out

linear forms

$$\begin{pmatrix} l_1 & f_1 \\ l_2 & f_2 \\ 0 & q \end{pmatrix}$$

conic