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n points $P_1, \dots, P_n \in \mathbb{C}^2$

$\binom{n}{2}$ lines L_{ij} where

L_{ij} connects P_i to P_j:

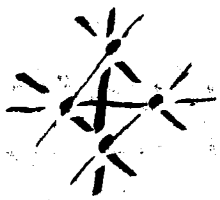
coordinates $x_1, y_1, \dots, x_n, y_n$ of points

& m_{ij} = slope of L_{ij} (affine piece $m_{ij} \neq \infty$)

$y_j - y_i = m_{ij}(x_j - x_i)$

$\Rightarrow$ pictures of the complete graph K_n on n points

K_4: generic part



dim $2n = 8$

degenerate: all points at same location

also dim 8

studied by Jeremy Martin

$V(K_n)$ = the graph variety = generic component

Thm $V(K_n)$ is cut out, set-theoretically, in the picture locus by the relations among the m_{ij}'s

i.e., by ideal $I(\Xi_n)$ of $p(m_{ij})$ s.t.

$$P\left(\left\{\frac{y_j - y_i}{x_j - x_i}\right\}\right) = 0.$$

S_n = slope variety

What is the algebraic dependence matroid of $\left\{\frac{y_i - y_j}{x_i - x_j}\right\}$

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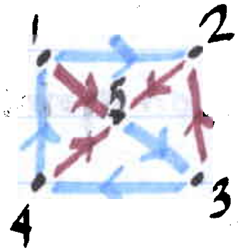
Thm The minimal dependent sets are the rigidity circuits: E s.t.

$|E| = 2|V(E)| - 2$, and for any F with $\emptyset \subsetneq F \subsetneq E$, $|F| \leq 2|V(F)| - 3$.

(This is known to be the algebraic dependence matroid of $\{(x_i - x_j)^2 + (y_i - y_j)^2\}$.)



Disjoint spanning trees S, T

$f \in S$, e.g., $f = \overrightarrow{34}$

Have unique coefficients $c_{ef} \in \{0, \pm 1\}$ s.t. $f - \sum_{e \in T} c_{ef}\, e$ is a directed cycle.

$\overrightarrow{34} - (\overrightarrow{32} + \overrightarrow{25} - \overrightarrow{45})$ is a directed cycle

$x_e = x_j - x_i \qquad y_e = y_j - y_i$

let $e = \overrightarrow{ij}$, $f = \overrightarrow{pq}$ then $x_q - x_p = \sum_{e \in T} c_{ef}\, x_e$

$$y_q - y_p = \boxed{m_{pq} \left(\sum_e c_{ef} x_e\right) = \sum_e c_{ef} m_e x_e}$$

$$\sum_e c_{ef} (m_{pq} - m_e) x_e = 0$$

the $(n-1)\times(n-1)$ matrix $M_{ST} = [c_{ef}(m_f - m_e)]_{e \in T, f \in S}$ has a nonzero nullvector

$$M_{ST} \begin{bmatrix} x_{e_1} \\ \vdots \\ x_{e_{n-1}} \end{bmatrix} = 0$$

$\Rightarrow \det M_{ST} \in I(S_n)$

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$$\begin{array}{cc} \begin{array}{c}53\\34\\41\\12\end{array} & \begin{bmatrix} 0 & m_{25}-m_{53} & m_{32}-m_{53} & 0 \\ 0 & m_{34}-m_{25} & m_{34}-m_{32} & m_{45}-m_{34} \\ m_{15}-m_{41} & 0 & 0 & m_{41}-m_{45} \\ m_{12}-m_{15} & m_{25}-m_{12} & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} 15 & 25 & 32 & 45 \end{array} \end{array}$$

Prop $D_{S\cup T}$ = det M_{ST} depends only on $S \cup T$ (up to sign)

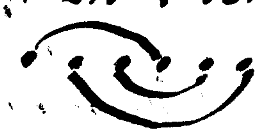
$D_{S\cup T} = \sum \pm$ (square-free monomials)

$D_{S\cup T}$ invertible $\Leftrightarrow$ $S \cup T$ is a rigidity circuit.

Thm) $\{D_W : W \text{ a wheel}\}$ is a Gröbner basis.

•) Initial ideal $\mathrm{in}(I(S_n))$ is Cohen-Macaulay (hence $I(S_n)$ is also), of dimension $2n-3$, and degree $(2n-5)(2n-7)\cdots 3\cdot 1$,

= # of matchings M_{2n-4} on $2n-4$ vertices



Open Question: Is it a universal Gröbner basis?

all rigidity circuits

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Prop Each $in(D_w)$ is the product $\prod_{e \in Q} M_e$
where Q is a Martin path

•5 endpts of Q, $\{x<y\}$ are the maximal 2 vertices (here 4,5)
4 $x - v - \cdots - w - y$
3
2 if $x<y$ then $v>w$
1

Prop Every maximal subgraph $H \leq K_n$ containing no Martin path has $2n-3$ edges; there are M_{2n-4} of them.

Def A simplicial complex Δ is Cohen-Macaulay (C-M) if the link of each face has just one nonzero homology group.
In particular, if Δ is shellable, it is Cohen-Macaulay.

Prop $\Delta = \{H \leq K_n : \text{no Martin path is contained in } H\}$ is shellable.

Thm (Hochster?) Δ is C-M iff R_Δ (Stanley-Reisner ring) is a C-M ring.

$$R_\Delta = k[x_1, \dots, x_d] / \left(\prod_{i \in N} x_i : N \notin \Delta\right)$$

now we have shown $J = (in(D_w) : W \text{ wheel})$ is C-M
$J = in(I(S_n))$

Prop $\dim S_n = 2n-3$, $\deg S_n \geq M_{2n-4}$

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now J is unmixed

$J \subseteq I$, $\dim J = \dim I$, $\deg J \leq \deg I$

$\Rightarrow I = J$. ■

$\mathcal{H}(q)$ = Hilbert series of $\mathbb{C}[\underline{m}]/I(S_n)$

$$= \frac{h(q)}{(1-q)^{2n-3}}$$

$h_n(q) = a_0 + a_1 q + a_2 q^2 + \cdots$, $a_i > 0$, $\sum a_i = h(1) = M_{2n-4}$

$= M_{2n-4}$

a_i = # of matchings on $[2n-4]$
with i "long" edges

0 long

1 long

2 long

$h_4(q) = 1 + q + q^2$