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$S = k[\underbrace{x_0, \ldots, x_r}_{W}] = \mathrm{Sym}\, W$

$E = \Lambda(\underbrace{W^*}_{V})$ exterior algebra $= \Lambda\langle \underbrace{e_0, \ldots, e_r}_{\text{dual to } x_i} \rangle$

<u>Prop</u> Linear free S-complexes $\overset{\mathbb{L}}{\leftarrow}\!\equiv$ graded E-modules

Graded S-modules $\equiv$ linear free E-complexes $\xleftarrow{\mathbb{R}}$

S and E are Koszul dual algebras

x_i's have deg 1, so functionals in W^* (e_i) have deg -1

P a graded E-module $P = \bigoplus_i P_i$

$\mathbb{L}P$:

$$\cdots \to S \otimes_k P_i \to S \otimes_k P_{i-1} \to S \otimes_k P_{i-2} \to \cdots$$

$$1 \otimes p \longmapsto \sum_j x_j \otimes e_j p \mapsto \sum_\ell \sum_j x_\ell x_j \otimes e_\ell e_j p$$

$$= \sum_{j \le \ell} x_\ell x_j \otimes (e_\ell e_j + e_j e_\ell) p = 0$$

so this is a differential, using commutativity in S & associativity & anticommutativity in E

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$M = \bigoplus M_i$ an S-module $\rightsquigarrow \mathbf{R}(M) \to E \otimes_k M_i \to E \otimes_k M_{i-1} \to \cdots$

write $\hat{P} = \mathrm{Hom}_k(P, k)$; say P a f.g. graded E-module

Prop 1) $L = \mathbf{L}(P)$ is

a subcomplex of the linear strand of a minimal free resolution

$\Leftrightarrow \hat{P}$ is ~~free~~ generated in degree 0, (by $\hat{P}_0$). $\quad S \otimes P_1 \to \cdots \to S \otimes P_0$

2) $L = \mathbf{L}(P)$ is

the linear strand of a minimal free res

$\Leftrightarrow \hat{P}$ is linearly presented

$E(1)^{a_1} \to E^{a_0} \to \hat{P}$

3) $L = \mathbf{L}(P)$ is

a free resolution $\Leftrightarrow \hat{P}$ has a linear free res.

$$(*)\ H_i(\mathbf{L}(P))_{i+d} = \mathrm{Tor}^E_d(\hat{P}, k)_{-i-d}$$

The case $P = E$ is just the Koszul complex

$\mathbf{R}(S) = \quad E \to E \otimes W \to E \otimes \mathrm{Sym}_2 W \to \cdots$

is the injective resolution of k as an E-module

this is the Cartan resolution,

cf. Cartan-Eilenberg

in general, for any graded module M,

$\mathbf{R}(M)$ is exact starting at reg M

the Castelnuovo-Mumford regularity of M.

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Given a finitely generated graded S-module M, what's the length of the linear strand of the min free resol'n of M? How long could it be?

Example 1 If M has 1 generator $m \in M_0$, then the length is $\dim\{w \in W \mid wm = 0\} = A_0$

Example 2 $S^\ell(-1) \xrightarrow{\varphi} S^2 \to N \to 0$

$$S = \Lambda^\ell S^\ell(-\ell) \otimes \mathrm{Sym}_0 S^2 \cdots \to \Lambda^2 S^\ell(-2) \otimes \mathrm{Sym}_{\ell-2} S^2 \to S^\ell(-1) \otimes \mathrm{Sym}_{\ell-1} S^2 \to \mathrm{Sym}_\ell S^2 \to M = \mathrm{Sym}_\ell N \to 0$$

has resolution of length ℓ

check case $\ell = 2$ $\quad 0 \to \Lambda^2 S^2(-2) \to S^2(-1) \otimes S^2 \to \mathrm{Sym}_2 S^2$

No element of M is annihilated by a linear form

M has $\ell+1$ generators

Let M be graded, $M = M_0 \oplus M_1 \oplus \cdots$, $M_0 \neq 0$

$A(M) = \{(x, \langle m \rangle) \in W \times \mathbb{P}(M_0) \text{ s.t. } xm = 0\}$

In Example 1: $\mathrm{ann}_W(m) \times \langle m \rangle$
w/ dimension = dim annihilator

In Example 2: $0 \times \mathbb{P}(M_0)$ has dim ℓ

Theorem (Mark Green)
$\dim A(M) \geq$ length of linear strand of resol'n of M

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other stronger conjectures are in
Eisenbud & Koh, 1991

suppose have linear $S \otimes P_i \xrightarrow{\varphi_i \ldots} S \otimes P_i$

$P_i \to W \otimes P_{i-1} \leftrightsquigarrow V \otimes P_i \to P_{i-1}$

Let L = linear strand = $\mathbb{L}(P)$

$\hat{P}$ is linearly presented $E^b(1) \to E^a \to \hat{P} \to 0$

L has length $\leq \dim A(M)$

$\iff (V)^{\dim A(M)+1}\, \hat{P} = 0$

annihilates — must show

1) find elements that annihilate a module
2) show you found a lot

try 1) $\hat{P}$ is annihilated by the "exterior minors" (Fitting ideal) of φ.

Case $\underline{b=1}$ 1 col presentation:

$$E(1) \xrightarrow[\begin{pmatrix} e_1 \\ \vdots \\ e_a \end{pmatrix}]{} E^a \to \hat{P} \to 0$$

p_i

$\sum e_i p_i = 0$

$\prod e_i$ annihilates · permanent of mtx obtained by repeating $\begin{pmatrix} e_1 \\ e_a \end{pmatrix}$ sufficiently

$(e_1 \cdots e_a) p_i = \pm (\prod_{j \neq i} e_j) e_i p_i = \pm (\prod_{j \neq i} e_j) \sum_{j \neq i} e_j p_j = 0.$