Dec. 6, '02

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"A generalization of tight closure and its applications"

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R Noeth. ring of prime char $= p > 0$, $q = p^e$

$R^\circ = R \setminus \bigcup_{\mathfrak{p} \in Min(R)} \mathfrak{p}$, if $I \subset R$ then

$I^{[q]} := (a^q : a \in I)$, $F^e : R \longrightarrow {}^eR$, $a \longmapsto a^q$

Def. $\mathfrak{a}, I \subset R$

The $\mathfrak{a}$-tight closure of I, $I^{*\mathfrak{a}}$, is

defined by $z \in I^{*\mathfrak{a}} \iff \exists\, c \in R^\circ$ s.t.

$c z^q \mathfrak{a}^q \subseteq I^{[q]}$ $\forall\, q = p^e \gg 0$

Can define the $\mathfrak{a}$-t.c. submodule $N \subseteq M$,

$N_M^{*\mathfrak{a}}$; in case $N = 0 \subset M$,

$z \in 0_M^{*\mathfrak{a}} \iff \exists\, c \in R^\circ$ s.t. $c\mathfrak{a}^q z^q = 0$ $\forall\, q \gg 0$

where

$$1 \otimes F^e : M = M \otimes_R R \longrightarrow M \otimes_R {}^e R = \mathbb{F}^e(M)$$
$$z \longmapsto z^q := z \otimes 1$$

(note $I^{*R} = I^*$ as usual)

Basic properties

- $I \subseteq I^{*\mathfrak{a}}$ ideal of R
- $J \subseteq I \Rightarrow J^{*\mathfrak{a}} \subseteq I^{*\mathfrak{a}}$
- $\mathfrak{b} \subseteq \mathfrak{a} \Rightarrow I^{*\mathfrak{b}} \supseteq I^{*\mathfrak{a}}$

Moreover if $\mathfrak{a} \cap R^\circ \neq \emptyset$ & $\mathfrak{b}$ is a reduction of $\mathfrak{a}$

$$\Rightarrow I^{*\mathfrak{a}} = I^{*\mathfrak{b}}$$

- $I^{*\mathfrak{a}\mathfrak{b}} \subseteq I^{*\mathfrak{a}} : \mathfrak{b}$

$$I^* \subseteq I^{*\mathfrak{a}} \subseteq I^* : \mathfrak{a}$$

Rmk. $I^* = (I^*)^* \subseteq \overline{I}$

but $I^{*\mathfrak{a}} \subsetneq (I^{*\mathfrak{a}})^{*\mathfrak{a}}$, $I^{*\mathfrak{a}} \nsubseteq \overline{I}$ in general

Def. $c \in R^{0}$ is an $\mathfrak{a}^{t}$-test element $\iff$

$c z^{q} \mathfrak{a}^{\lceil qt \rceil} \subseteq I^{[q]}$ for $\forall q$, whenever $z \in I^{*\mathfrak{a}}$

$(t \in \mathbb{Q}_{>0})$.

Prop.-Def. $E = \bigoplus_{m \in \mathrm{Max}(R)} E_R(R/m)$

$\tau(\mathfrak{a}) = \bigcap_{M: \text{f.g. } R\text{-mod.}} \mathrm{Ann}_R(0_M^{*\mathfrak{a}}) = \bigcap_{I \subseteq R} I : I^{*\mathfrak{a}}$

R: loc. approx. Gor.

$= \mathrm{Ann}_R(0_E^{*\mathfrak{a}})$

R: excellent normal $\mathbb{Q}$-Gor.

(also, can replace $\mathfrak{a}$ with $\mathfrak{a}^{t}$)

Rmk. • $\mathfrak{b} \subseteq \mathfrak{a} \Rightarrow \tau(\mathfrak{b}) \subseteq \tau(\mathfrak{a})$

If $\mathfrak{a} \cap R^{\circ} \neq \emptyset$ & $\mathfrak{b}$: red. of $\mathfrak{a}$ $\Rightarrow$

$\tau(\mathfrak{b}) = \tau(\mathfrak{a})$

• $\tau(\mathfrak{a}) \mathfrak{b} \subseteq \tau(\mathfrak{a}\mathfrak{b})$

Rmk. $\mathfrak{a} = R$

$\tau(R) \cap R^\circ = \{ (R\text{-})\text{test elements of } R \}$

$\mathfrak{a} \neq R$

$\tau(\mathfrak{a}) \cap R^\circ \subsetneq \{ \mathfrak{a}\text{-test elt. of } R \}$

Prop. R: F-finite, normal $\mathbb{Q}$ Gor.

$\Rightarrow \forall$ test elt. of R is an

$\mathfrak{a}^t$-test element $(\forall \mathfrak{a} \subseteq R, \forall t \geq 0)$

$\tau(\mathfrak{a}) \cap R^\circ \subseteq \tau(R) \cap R^\circ \subseteq \{ \mathfrak{a}^t\text{-test elt.} \}$

Ex. (R, m) d-dim RLR $\Rightarrow$

$$\tau(m^n) = \begin{cases} m^{n-d+1} & \text{if } n \geq d \\ R & \text{if } 0 \leq n \leq d-1 \end{cases}$$

$\tau(m^{d-1}) = R \Rightarrow R$: regular

$\uparrow$

$\exists$ cond.

<u>Thm.</u> $t \in \mathbb{Q}_{\geq 0}$ fix (R, m) normal $\mathbb{Q}$-Gor.

e.f.t. $/k$, $\operatorname{char}(k) = 0$.

$f : X \longrightarrow \operatorname{Spec}(R)$ a log-resolution

of an ideal $\mathfrak{a} \subseteq R$, i.e. $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-Z)$,

$\operatorname{Exc}(f) \cup \operatorname{Supp}(Z)$: SNC divisor.

$K_{X/R} = K_X - f^* K_R$

Then in reduction modulo $p \gg 0$,

$$\tau(\mathfrak{a}^t) = H^0(X, \mathcal{O}_X(\lceil K_{X/R} - tZ \rceil)) \quad \text{in } R.$$

$$\tau(\mathfrak{a}^t) = \operatorname{Ann}_R(0_E^{*\mathfrak{a}^t}), \qquad H^0(X, \mathcal{O}_X(\lceil K_{X/R} - tZ \rceil)) = \mathcal{J}(\mathfrak{a}^t) \text{ multiplier ideal}$$

$d = \dim(R)$, in char $= p \gg 0$

$$\operatorname{Ann}_{\omega_R}(0^{*\mathfrak{a}^t}_{H^d_m(R)}) = H^0(X, \omega_X(\lceil -tZ \rceil)) \quad \text{in } \omega_R$$

$$\operatorname{Ann}_{\omega_R}(0^{*\mathfrak{a}^t}_{H^d_m(R)}) = \tau(\mathfrak{a}^t \cdot \omega_R), \qquad H^0(X, \omega_X(\lceil -tZ \rceil)) = \operatorname{adj}(\mathfrak{a}^t \cdot \omega_R)$$

char p $\xleftarrow[\text{mod } p \gg 0]{\text{reduction}}$ char 0

$\tau(\mathfrak{a}^t)$ $\qquad\qquad$ $\mathcal{J}(\mathfrak{a}^t)$

? $\qquad\qquad$ restriction thm.

? $\qquad\qquad$ subadditivity (Demailly–Ein–Lazarsfeld)

? $\qquad\qquad$ Skoda's thm. (Lipman)

(tight closure Briançon–Skoda) $\Rightarrow$ Briançon–Skoda thm. $\Leftarrow$

From now on everything in char $= p > 0$

<u>Thm.</u> (restriction thm.)

$(R, \mathfrak{m})$ normal $\mathbb{Q}$-Gor. complete local

$\mathfrak{a} \subset R$, $0 \neq x \in \mathfrak{m}$ s.t. $S = R/xR$

normal.

$$\tau(\mathfrak{a}S^t) \subseteq \tau(\mathfrak{a}^t)S$$

Thm. (Subadditivity)

$(R, \mathfrak{m})$ complete RLR ; $\mathfrak{a}, \mathfrak{b} \subset R$; $t, t' \geq 0$

$\tau(\mathfrak{a}^t \mathfrak{b}^{t'}) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^{t'})$.

Thm. (Skoda-type thm.)

$(R, \mathfrak{m})$ complete or normal & $\mathbb{Q}$-Gorenstein.

$\mathfrak{a} \subseteq R$ ideal of ht > 0 with

reduction gen. by l elts. $\Longrightarrow$

$\tau(\mathfrak{a}^l) = \tau(\mathfrak{a}^{l-1})\, \mathfrak{a}$

Pf. $\supseteq$ okay ✓

$\subseteq$ may assume $\mathfrak{a}$ is gen. by l elts.

$\mathfrak{a}^{ql} = \mathfrak{a}^{q(l-1)} \mathfrak{a}^{[q]}$

claim: $0_E^{*\mathfrak{a}^l} = 0_E^{*\mathfrak{a}^{l-1}} : \mathfrak{a}$ in $E = E_R(R/\mathfrak{m})$

$$\begin{array}{c} \cup \\ z \end{array}$$

$z\mathfrak{a} \subseteq 0_E^{*\mathfrak{a}^{l-1}}$

$c z^q \overbrace{\mathfrak{a}^{[q]} \mathfrak{a}^{q(l-1)}}^{\mathfrak{a}^{ql}} = 0 \quad \forall q \gg 0$ ✓

$$\tau(\mathfrak{a}^l) = \mathrm{Ann}_R\left(0_E^{*\mathfrak{a}^l}\right) = 0_E^{*\mathfrak{a}^{l-1}} : \mathfrak{a}$$

$$\parallel \; \left(0 :_E \tau(\mathfrak{a}^{l-1})\right)$$

Q-Gor.

$$= \mathrm{Ann}_R\left(0 :_E \tau(\mathfrak{a}^{l-1})\, \mathfrak{a}\right) = \tau(\mathfrak{a}^{l-1})\mathfrak{a}.$$

Cor. $(R, \mathfrak{m})$ as in thm.

① R : weakly F-reg. $\Rightarrow \overline{\mathfrak{a}^{n+l-1}} \subset \mathfrak{a}^n \quad \forall n \gg 0$

Briançon–Skoda

② If $\dim R = d$ & $R/\mathfrak{m}$ infinite

$$\tau(\mathfrak{a}^{n+d-1}) = \tau(\mathfrak{a}^{d-1}) \cdot \mathfrak{a}^n \quad \forall n \gg 0$$

②' $\tau(\mathfrak{a}^{n+d-1}\omega_R) = \tau(\mathfrak{a}^{d-1}\omega_R) \cdot \mathfrak{a}^n$ in ω_R $\forall n \gg 0$

Thm. (K.E. Smith)

X : proj variety $/ k = \bar{k}$, char $k = p > 0$
with only F-rat'l sings.

$\mathcal{L}$: glob. generated ample inv. sheaf on X

$\Rightarrow \omega_X \otimes \mathcal{L}^{\otimes \dim X + 1}$: glob. gen.

<u>outline of pf.</u> $d = \dim R = \dim X + 1$

$R = R(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$ graded ring

$\exists$ homog. s.o.p. of deg 1

① $\omega_X \otimes \mathcal{L}^d$: gl. gen. $\Longrightarrow$ $[\omega_R]_n = R_{n-d}[\omega_R]_d$ $\forall n \gg 0$

$\omega_R = \bigoplus_{n \in \mathbb{Z}} H^0(X, \omega_X \otimes \mathcal{L}^{\otimes n})$

② in general $\tau(m^{n-1} \cdot \omega_R) \subseteq [\omega_R]_{\geq n}$

and if $n \gg 0$ equality holds.

③ $\tau(m^{n+d-1} \omega_R) = \tau(m^{d-1} \omega_R) m^n$ $\forall n \gg 0$.

<u>$n \gg 0$</u> $R_{n-d} \cdot [\omega_R]_{\geq d} \subseteq [\omega_R]_{\geq n} \overset{②}{=} \tau(m^{n-1} \omega_R)$

$\overset{③}{=} \tau(m^{d-1} \omega_R) m^{n-d} \subseteq R_{\geq n-d} [\omega_R]_{\geq d}$

$\therefore R_{n-d} [\omega_R]_{\geq d} = [\omega_R]_{\geq n}$

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