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"Non-vanishing of cohomology"

$(R, \mathfrak{m}, k)$ comm. Noeth. local ring

$\dim R$ = Krull-dim. of R
$\operatorname{edim} R$ = embedding-dim. of R
$\operatorname{codim} R = \operatorname{edim} R - \dim R$

$M, N \in \operatorname{mod}(R) = \{\text{finite } R\text{-modules}\}$

$$\tilde{P}(R) = \{ M : \operatorname{pd}_R(M) < \infty \}$$

$$\tilde{I}(R) = \{ M : \operatorname{id}_R(M) < \infty \}$$

M $\quad M_T^{\perp} = \{ N \in \operatorname{mod}(R) : \operatorname{Tor}_n^R(M,N) = 0 \ \forall n \gg 0 \}$

$\quad M_E^{\perp} = \{ \quad \text{"} \quad : \operatorname{Ext}^n(M,N) = 0 \quad \text{"} \quad \}$

$\quad {}_E M^{\perp} = \{ \quad \text{"} \quad : \operatorname{Ext}^n(N,M) = 0 \quad \text{"} \quad \}$

<u>Rmk.</u> $\quad M_T^{\perp} \supseteq \tilde{P}(R)$

$\quad {}_E^{\perp}M \quad \text{"}$

<u>Thm.</u> (Buchweitz, _) If R is c.i.

then for each M:

$$\underbrace{M_T^{\perp} = M_E^{\perp} = {}_E M^{\perp}}\quad (= M^{\perp})$$

$\Downarrow \quad \Uparrow ?$

<u>Rmk.</u> R is Gorenstein

Known: $M_E^{\perp} = {}_E M^{\perp}$ does not characterize c.i.

(Huneke, Jorgensen): All Gor. rings of min. mult. have it

(Şega)

$$cx_R(M) = \inf\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists\, a \in \mathbb{R} \text{ s.t.} \\ \operatorname{rank}_k \operatorname{Tor}_n^R(M,k) \le a n^{d-1} \ \forall n \gg 0 \end{array} \right\}$$

$cx_R(M) = 0 \iff pd_R(M) < \infty$

$cx_R(M) = \infty$ is possible, but if R is c.i.

then:

<u>Thm.</u> (Shamash) $\operatorname{cx}_R M \leq \operatorname{codim} R$

R is c.i. if $R \to \hat{R} \xleftarrow[\text{seq.}]{\text{reg.}} \tilde{R}$

w.l.o.g. $\operatorname{edim} \tilde{R} = \operatorname{edim} R$

<u>Thm.</u> if R is c.i. then

$$M^{\perp} = \tilde{P}(M) \iff \operatorname{cx}_R M = \operatorname{codim} R$$

(in fact, vanishing of $c+1$ consecutive Ext's or Tor's suffices.)

Structures on cohomology

$\mathcal{E} = \operatorname{Ext}_R^{\cdot}(k,k)$ grade algebra w.r.t. Yoneda pdcts.

$$0 \to k \to X_{n-1} \to \cdots \to X_0 \to k \to 0$$

$$0 \to k \to Y_{m-1} \to \cdots \to Y_0 \to M \to 0$$

$\operatorname{Ext}_R^{\cdot}(M,k)$ is a graded left $\mathcal{E}$-module.

Structure of $\mathcal{E}$

$$\mathrm{Ext}^{\cdot}_{R}(k,k) \cong \mathrm{Ext}_{\tilde R}(k,k) \twoheadrightarrow \overbrace{\underbrace{\mathrm{Ext}_{\tilde R}(k,k)}_{=\,\Lambda(m/m^2)^*}}^{\Lambda\,=} \to 0$$

Serre's pf. that $pd_R k < \infty \Rightarrow R$ regular.

$\mathcal{R}$ is a graded normal sub-algebra of $\mathcal{E}$.

$\hookrightarrow \mathcal{R}^+\mathcal{E} = \mathcal{E}\mathcal{R}^+$

s.t. $\mathcal{E}/\mathcal{R}_+\mathcal{E} \cong \Lambda$ $\qquad \mathcal{R} \hookrightarrow \mathcal{E} \to \Lambda$

~~Structure~~

<u>Fact</u> If R is c.i. then $\mathcal{R}$ is a polynomial algebra gen. by $c = \text{codim } R$ vars of deg 2 and $\mathcal{R}$ is central in $\mathcal{E}$.

<u>Ex.</u> If $m^2 = 0$ then $\mathcal{E}$ is the tensor algebra on $(m/m^2)^*$

<u>Thm.</u> (Gulliksen) if R is c.i. then

$\mathcal{M} = \mathrm{Ext}_R(M,k)$ is f.g. $/\mathcal{R}$ ($\Leftrightarrow$ over $\mathcal{E}$).

Support variety (scheme) of M

$$\overline{V}_R(M) = \mathrm{Proj}\left(\mathcal{R}/\mathrm{ann}_{\mathcal{R}}\mathcal{M}\right)$$

$$V_R^*(M) = V\left(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})\right) \subseteq k^c \qquad (c = \mathrm{codim}\, R)$$

<u>Properties</u> (0) $\overline{V}_R(M) = \phi \Leftrightarrow \mathrm{pd}_R(M) < \infty$

(1) $\dim \overline{V}_R(M) = \mathrm{cx}_R(M) - 1$

(2) $\overline{V}_R(M) \cap \overline{V}_R(N) = \phi \Leftrightarrow \mathrm{Ext}_R^n(M,N) = 0 \quad \forall n \gg 0.$

<u>DGHA</u>

A DG algebra A is a complex

along with morphisms:

$\mu : A \otimes A \longrightarrow A$

$\eta : R \longrightarrow A$

s.t. μ is associative and η is unit.

$\bigoplus_{i \in \mathbb{Z}} A_i$, $A \otimes B \to B$

Note: DG-module over $R \equiv$ cx. of R-modules.

<u>Ex.</u> K = Koszul complex on a min'l set of gens. $(t_1, \ldots, t_\ell)$ of m.

$K = \Lambda(R^\ell)$ as a graded algebra is a DG with the usual diff $x_1, \ldots, x_\ell \in K_1$, $\partial(x_i) = t_i$.

A DG-module B/K is just a cx. of R-modules along with deg 1 maps

$B_n \xrightarrow{\cdot x_i} B_{n+1}$ which are homotopie

$t_i 1_B \sim 0_B$

<u>Fact</u> For every DG-module B over A $\exists$ quasi-isom.

$$\begin{tikzcd} P' \arrow[d, dashed, shift right] \arrow[dr, "\simeq"] & \\ P \arrow[u, shift right] \arrow[r, "\simeq"] & B \end{tikzcd}$$

where P is a semi-free ($\frac{1}{2}$ free) A-module.

In case A is non-neg. graded and B is bounded below, then every DG module P which is free after forgetting the diff. is $\frac{1}{2}$-free.

<u>Note</u> $K \to k$ is a morph. of DG algs, so k is a DG module.

$$\begin{array}{l} \operatorname{Ext}_K(k,k) \\ \quad \cong \\ H\operatorname{Hom}_A(P,P) \end{array}$$

This is actually a graded algebra

$P \xrightarrow{\simeq} k$

Warning: this composition pdct differs from Yoneda's pdct by a sign.

Thm. $\mathrm{Ext}_K(k,k) \simeq \mathcal{R}$

$$R \longrightarrow \hat{R} \xleftarrow{\simeq} \tilde{R}$$

$\hookrightarrow$ recall: RLR!

$$K \to \hat{K} = \hat{R} \otimes_{\tilde{R}} \tilde{K} \xleftarrow{\simeq} E \otimes_{\tilde{R}} \tilde{K} \xrightarrow{\simeq} E \otimes_{\tilde{R}} k$$

Let E be a DG algebra over $\tilde{R}$ s.t.

$H(E) = \hat{R}$.

(If R is c.i., $E =$ Koszul cx. on a min. set of relations of $\hat{R}$.)

$$\tilde{R} \longrightarrow \hat{R} \longrightarrow \hat{K} \quad \text{(fibration sequence)}$$

$$\begin{array}{ccccc} \mathrm{Ext}_{\tilde{R}}(k,k) & \longleftarrow & \mathrm{Ext}_{\hat{R}}(k,k) & \longleftarrow & \mathrm{Ext}_{\hat{K}}(k,k) \\ \| & & \| & & \| \\ \Lambda & & \mathcal{E} & & \mathcal{R} \end{array}$$

<u>Thm.</u> (Jorgensen, —)

If M is a DG module over K s.t. $H(M)$ is of finite rank over k, then

$$\mathrm{Ext}_R(k,k) \otimes_{\mathrm{Ext}_K(k,k)} \mathrm{Ext}_K(M,k) \cong \mathrm{Ext}_R(M,k)$$

$$\mathcal{E} \otimes_{\mathcal{R}} \mathrm{Ext}_K(M,k) \cong \mathcal{M}$$

<u>Thm.</u> For every $\mathcal{R}$-mod. (graded) $\mathcal{M}'$

$\exists$ f.g. R-module M' s.t.

$$\mathrm{Ext}(M',k)$$

<u>Cor.</u> For every proj. sub-scheme of $\mathbb{P}_k^{c-1}$

$\exists$ a f.g. R-mod. M s.t. $V_R(M) = V$.