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"Core and integral closure of ideals"

§1 Notations & definitions

(R, m, k) Gor. $|k| = \infty$ $d = \dim R$

$I \subset R$ ideal, $J \subset I$ a reduction of I,

i.e. $\exists n \geq 0$ $I^{n+1} = JI^n$, or equiv.

$R[Jt] \hookrightarrow R[It]$ integral.

$J \in m(I)$, minimal reduction

$r_J(I) :=$ smallest n s.t. $I^{n+1} = JI^n$

$r = r(I) := \min \{ r_J(I) : J \in m(I) \}$

$\ell = \ell(I) := \mu(J) = \dim R[It] \otimes K$ $\forall J$

$\hookrightarrow$ analytic spread

$g = ht\, I > 0$, $g \leq \ell \leq d$

$J \subset I \subset \bar{I}$

Rees/Sally $\quad \mathrm{core}(I) := \bigcap_{J \in \mathcal{R}(I)} J$

§2 Motivations

- uniform properties of reductions
- Briançon - Skoda RLR

$$\overline{I^\ell} \subset \mathrm{core}(I)$$

$$\cap \quad \mathrm{adj}(I^\ell) \quad \cup \rightsquigarrow \text{Lipman}$$

- Kawamata conjecture

§3 Assumptions

1. I G_ℓ : $\mu(I_p) \le \dim R_p \quad \forall\, p \in V(I)$
   $\dim R_p \le \ell - 1 < d$

2. $\operatorname{depth} R/I^j \ge \dim R/I - j + 1 \qquad 1 \le j \le \ell - g$

Rmk
- m-primary ideals
- equimultiple ideals $(\ell = g)$
- 1-dim g.c.i.

- CM g.c.i. with $\ell = g+1$

If I G_ℓ licci $\Rightarrow$ SCM $\Rightarrow$ 2

§4 History

Huneke - Swanson: RLR, $\dim = 2$, $I = \overline{I}$

$$\Rightarrow \operatorname{core}(I) = (J:I)I = \operatorname{adj}(I^2) \quad \forall J \in m(I)$$

C.P.U. . core is a finite intersection of general minimal reductions.

. core behaves well under flat extensions

. TFAE

a. $\operatorname{Core}(I) = (J:I)I = (J:I)J \quad \forall J \in m(I)$
b. $I(J:I) \subset \operatorname{core}(I)$ for some J.

c. $J:I$ does not depend on J

d. $r \leq \ell - g + 1$

Conjecture: $\operatorname{core}(I) = J^{r+1} : I^r =$

$$J(J^r : I^r) = I(J^r : I^r)$$

where $r = r_J(I)$.

Proved by Hyry - Smith R, $R[It]$ CM, $\mathrm{char}(k) = 0$, ~~and~~ and I equimultiple

Huneke - Trung R CM, $\mathrm{char}(k) = 0$, I equimult.

§5 A formula for the core

Thm (P, Ulrich)

$$J^{n+1} : I^n \subset \mathrm{core}(I) \subset J^{n+1} : \sum_{y \in I} (J, y)^n$$

$\forall\ n \geq \max\{r_J(I) - \ell + g, 0\}$

In part. if

- $\mu(I) \leq \ell + 1$ or
- $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > r - \ell + g$ then

$$\mathrm{core}(I) = J^{n+1} : I^n$$

$\ell = 1$ $\quad R$ CM

$\ell = g = 1$ $\quad$ let $t = \max\{e(I_p) : p \in \mathrm{Min}(I)\}$

then $\mathrm{core}(I) = J^{n+1} : \sum_{y \in I_t} (J, y)^n$

$= J^{n+1} : \sum_{i=1}^{t} (J, y_i)^n$

$\forall\, n \geq s = \max\{ r((J, y_i)) : 1 \leq i \leq t \}$

note $\sum_{y \in I} (J, y)^n$ ——— I^n in $ch = 0$

or $ch = k > r$.

<u>EX.</u> $R = \dfrac{k[x,y,z]_{(x,y,z)}}{(y^2, z^2)}$ $\quad I = m = (x, y, z)$

$J = (x)$ then $r_J(m) = 2$

$\mathrm{core}(m) = J^3 : m^2$ in $ch = 0$

$ch(k) > 2$.

In $\mathrm{char}(k) = 2$

$r((J, y_i)) = 1$ $\quad \mathrm{core}(m) = J^2 : \sum_{y \in m} (J, y)$

$= J^2 : m = J^3 : xm \neq J^3 : m^2$

§6 the canonical module

thm. 2 (P, Ulrich)

$$\omega_{R[It,t^{-1}]} = \bigoplus_{j\in\mathbb{Z}} (J^j : I^{i-\ell+g})\, t^{j-i+\ell-1} \subset R[t,t^{-1}]$$

$\forall\ i \geq \max\{r_J(I), \ell-g\}$

sketch: let $A = R[Jt, t^{-1}]$

$B = R[It, t^{-1}]$

By Ulrich $\omega_A = (1, Jt)^{\ell-g} A\ t^{g-1}$

let $L = (1, (1, It)^{\ell-g} I^{i-\ell+g} t^{i-\ell+g}) A$

$\subset R[t, t^{-1}]$

$0 \to L \to B \to C \to 0$

$\operatorname{grade}_A C \geq 2 \Rightarrow$

$\omega_B \simeq \operatorname{Hom}_A(B, \omega_A) \simeq \operatorname{Hom}_A(L, \omega_A)$

$\cong$

$(A :_{R[t,t^{-1}]} I^{i-\ell+g})\, t^{\ell-i+1}$

$\forall\ n \geq \max\{r_J(I) - \ell + g, 0\}$

$$\left.\begin{array}{l} J^{n+1} : I^n \text{ is indep. of } J \\ J^{n+1} : I^n \subset J^{n+1} : J^n = J \end{array}\right\} \Longrightarrow$$

$J^{n+1} : I^n \subset \operatorname{core}(I)$.

$\operatorname{char}(k) = 0 \qquad \operatorname{char}(k) > r - \ell + g$

$\operatorname{core}(I) = [\omega_{R[It, t^{-1}]}]_g$

When is $\operatorname{core}(I) = J\underbrace{(J^n : I^n)}_{[\omega]_{g-1}}$

- $gr_I(R)$
- I equimultiple & S_2-ification $R[It, t^{-1}]$ CM
- " & R is a domain e.f.t. / field of ch = 0 and Proj $R[It]$ is smooth.

$J^{j+n} : I^n = J^j (J^n : I^n)$

When is $\operatorname{core}(I)$ i.c. ? (i.c. = integ. closed)

Cor. 3 (P, Ulrich)

R normal, $ch(k) = 0$ or $> r - \ell + g$

$Proj(R[It])$ has R_1 $\Rightarrow$

$core(I) = \overline{core(I)}$

Pf. $A = R[Jt, t^{-1}]$, $B = R[It, t^{-1}]$

$D = \bigoplus_j (J^{n+j+1} : I^n) t^j \quad \forall n \geq \max\{r - \ell + g, 0\}$

– $D \subset A \subset \overline{B}$

– $[D]_0 = core(I)$

– $\overline{B}/B$ has grade ≥ 2

$\Downarrow$

$\omega_{\overline{B}} \cong \omega_B \cong D \Rightarrow \underset{\substack{\parallel \\ core(I)}}{[D]_0}$ is i.c.

in $[\overline{B}]_0 = R$

Cor 4 (P, Ulrich) R RLR e.f.t. $/k$, $char(k) = 0$

$R[It]$ has only rational sing. $\Rightarrow core(I) = adj(I^g) = I\, adj(I^{g-1})$

Ex. (—, Ulrich, Vitulli)

$k[x,y]$

I gen. by monomials of the same degree.

$I = c\,(x^a, x^{b_1}y^{a-b_1}, x^{b_2}y^{a-b_2}, \ldots, x^{b_s}y^{a-b_s}, y^a)$
$\hookrightarrow$ monomial

let $\delta = \gcd(a, b_1, \ldots, b_s)$

then $\mathrm{core}(I) = c\,(x^\delta, y^\delta)^{2\frac{a}{\delta}-1}$

W.l.o.g. $c=1$

w.l.o.g. $\delta = 1$ $\quad \mathrm{core}(I) = \mathrm{core}(m^a)$

$= m^{2a-1} = \mathrm{adj}(I^2)$

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