Orlando E. Villamayor Dec. 2, '02

"On hypersurface singularities & multiplicity"

$f: Z^2 + XY^2 \in k[X, Y, Z] = S$, $char(k) = 0$.

$V(f) = X \subset \mathbb{A}^3_k = Spec(S)$.

Find all points of mult. 2.

- (R, M) loc. reg., J has order b if
$$J \subset M^b, \quad J \not\subset M^{b+1}$$
- $p \in V(f) = X$. X has mult. b at the point p if $\langle f \rangle$ has order b at S_p, or p is a b-fold point.

1) $J = \langle f \rangle$, $J_1 = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$V(J_1) \subset V(f) = X$ points of mult. 2 (or more).

2) i) $Z^2 + aZ + b$ $\quad a, b \in k[X, Y]$

$$\frac{\partial f}{\partial Z} = 2Z + a, \qquad a = 0, \ char(k) \neq 2$$

$Z \in J_1$, $V(J_1) \subset V(Z) = H$ smooth hypersurface

2) ii) $\{ P \in \mathrm{Spec}(S) \mid \text{order of } f \text{ on } S_P \text{ is } \geq 2 \} =$

$\{ P \in \mathrm{Spec}(S) \mid Z \in P \text{ and order of } XY^2 \geq 2 \text{ at } S_P \}$

$H = V(Z)$. $\langle XY^2 \rangle \subset \mathcal{O}_H$

$= \{ x \in H \mid \langle XY^2 \rangle \text{ has order } \geq 2 \text{ at } \mathcal{O}_{H,x} \}$

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2) i) $F = V(J_1) \subset H$ smooth hyp.

ii) F is described by $((XY^2), 2)$

$\langle XY^2 \rangle \subset \mathcal{O}_H$.

- Tschirnhausen
- Generic projection

$$\mathbb{A}^3 \xleftarrow[\pi]{\text{blow up } x_0} W$$

$X \qquad\qquad X_1$ s.t.

$H \qquad\qquad H_1$ s.t. $\qquad H \xleftarrow[E]{} H_1$ bl. up at x_0

$\qquad\qquad\qquad\qquad x_0$

$F \ni x_0 \qquad\qquad \langle XY^2 \rangle \mathcal{O}_{H_1} = I(E)^2 \mathcal{J}_1$

Claim: i) the set of $\overset{2}{b}$-fold pts. in $X_1 \subset H_1$

ii) the set of $\overset{2}{b}$-fold pts. is described by $(\mathcal{J}_1, 2)$

$$k[X,Y,Z] \hookrightarrow k[\tfrac{X}{Y}, Y, \tfrac{Z}{Y}]$$

$$Z^2 + XY^2 \qquad \left(\tfrac{Z}{Y}\right)^2 + \left(\tfrac{X}{Y}\right) Y^2$$

$$\left(\tfrac{Z}{Y}\right)^2 + a\left(\tfrac{Z}{Y}\right) + b \; ; \qquad a, b \in k[\tfrac{X}{Y}, Y]$$

~~$V(\tfrac{Z}{Y}) \in$ set of points of mult. 2~~

2i) $V(\tfrac{Z}{Y}) = H_1^{(1)} \supset$ set of pts. of mult. 2

2ii) In $V(\tfrac{Z}{Y}) = H_1^{(1)}$ the set of pts. of mult. 2 is defined

$$\mathcal{O}_{H_1^{(1)}} \left(\left(\tfrac{X}{Y} \cdot Y\right), 2 \right)$$

$$H \longleftarrow H_1^{(1)}$$

$$xy^2 \qquad y^2\left(\tfrac{x}{y}\, y\right)$$

$$k[X,Y] \longrightarrow k\left[\tfrac{X}{Y}, Y\right]$$

$f \in k[X_1, \dots, X_n]$ $\qquad V(f) = X \subset \mathbb{A}^n_k$

$J = \langle f \rangle$. $\qquad J(b-1) = \left\langle f, \dfrac{\partial f}{\partial X_1^{\alpha_1} \cdots \partial X_n^{\alpha_n}} \;\middle|\; \sum_i \alpha_i \leq b-1 \right\rangle$

$z^b + a_1 z^{b-1} + \dots + a_b$, $\qquad z_1 = z + \frac{1}{b} a_1 \in k[z_1, X_2, \dots, X_n]$

$a_i \in k[X_2, \dots, X_n]$

$z_1^b + a_2' z_1^{b-2} + \dots + a_b'$ $\qquad z_1 \in J(b-1)$

$F(b) \subset V(z_1) = H$ smooth hyp.

b-fold pts.

$$F(b) = \left\{ x \in H \;\middle|\; \mathcal{O}_{H,x} \quad \begin{matrix} \text{order of} \\ a_2' \geq 2 \\ a_3' \geq 3 \\ \vdots \\ a_b' \geq b \end{matrix} \right\}$$

$$J = \left\langle (a_2')^{\frac{b!}{2}}, (a_3')^{\frac{b!}{3}}, \dots, (a_b')^{\frac{b!}{b}} \right\rangle$$

i) $F(b) \subset H$

ii) described by $(J, b!)$.

$\mathbb{A}^n_k \xleftarrow{\text{bl. up } Y} W \qquad\qquad H \xleftarrow{\text{bl. up } Y} H_1$

$X \qquad\qquad X_1$ s.t.

$H \qquad\qquad H_1$ s.t.

$$\begin{array}{c} F(b) \\ \cup \\ Y \end{array}$$

$$J\mathcal{O}_H = I(E)^{b!} J_1$$

P1) b-fold pts. of $X_1 \subset H$

P2) b-fold pts. of X_1 described $(J_1, b!)$

$X \subset \mathbb{A}^n_k$ hyp. surface. $\quad x_0 \in X$ closed b-fold pt.

$\mathcal{O}_{\mathbb{A}^n_k, x_0} \qquad \hat{\mathcal{O}}_{\mathbb{A}^n_k, x_0}$

Weierstrass: $\quad f = z^b + a_1 z^{b-1} + \cdots + a_b \in R[z]$

$R = k[[x_2, \ldots, x_n]]$

R reg. $\quad B = R[z]/\langle z^b + a_1 z^{b-1} + \dots + a_b \rangle$

$a_i \in R$

$$\begin{array}{l} \hat{X} = \operatorname{Spec}\left(R[z]/\langle z^b + a_1 z^{b-1} + \dots + a_b \rangle \right) \\ \downarrow \\ \operatorname{Spec}(R) \end{array} \qquad \begin{array}{c} B \\ \uparrow \\ R \text{ reg.} \end{array}$$

Zariski: $\quad R \quad M \qquad K = R_M/M$

$$\begin{array}{ccc} & M & \\ \diagup & & \diagdown \\ M_1, & \dots, & M_s \end{array}$$

$B \quad M_1, \dots, M_s \qquad K_i = B_{M_i}/M_i B_{M_i}$

$$b = \sum_{i=1}^{s} \dim_K (K_i) \cdot e_{M B_{M_i}}(B_{M_i})$$

<u>Proposition 1</u> $\quad$ $\hat{X} = \operatorname{Spec}(B) \qquad F(b) \subset \hat{X}$ b-fold pts.

$$\begin{array}{ccc} \hat{X} = \operatorname{Spec}(B) & & F(b) \subset \hat{X} \\ \pi \downarrow \quad \downarrow & & \downarrow \pi \\ W \quad \operatorname{Spec}(R) & & \pi(F(b)) \subset W \end{array}$$

Given $x \in \pi(F(b))$, $\pi^{-1}(x) = \{y\}$ and the residue fields are equal.

$e_{M_i B_{M_i}}(B_{M_i}) \le e(B_{M_i}) = b \qquad s = 1$ and $K_1 = K$.

<u>Proposition 2</u>

$$\begin{tikzcd} F(b) \; \hat{X} = \mathrm{Spec}(B) \arrow[d, "\pi"'] \\ \pi(F(b)) \; W \subset \mathrm{Spec}(R) \end{tikzcd}$$

1) if $Y \subset \pi(F(b)) \subset W$ is smooth & closed
$\exists ! \; \tilde{Y} \subset F(b)$ mapping to Y and $\tilde{Y} \xrightarrow{\sim} Y$

2)
$$\begin{tikzcd} \hat{X} \arrow[d, "\pi"'] & X_1 \arrow[l, "\text{bl. up } \tilde{Y}"'] \arrow[d, "\pi_1"] \\ W & W_1 \arrow[l, "\text{bl. up } Y"'] \end{tikzcd}$$

π_1 is also a simple cover of degree b.

<u>Thm.</u> ($\mathrm{char}(k) = 0$) To each simple cover of deg. b $\xrightarrow{\text{assign}}$ a pair

$$\begin{tikzcd} \hat{X} \arrow[d, "\pi"'] \\ W = \mathrm{Spec}(R) \end{tikzcd}$$

(J, b^*), $\quad J \subset \mathcal{O}_W$.

P1) $\pi(F(b))$ described by (J, b^*)

P2)

$$P2)\quad F(b) \subset \hat{X}$$

$$\begin{array}{c} F(b) \subset \hat{X} \\ \big\downarrow \pi \\ \pi(F(b)) \quad W \end{array}$$

$Y \subset \pi(F(b))$

$$\begin{tikzcd} \hat{X} \arrow[d] & X_1 \arrow[l] \arrow[d, "\pi_1"] \\ W & W_1 \arrow[l] \end{tikzcd}$$

as in Prop. 2

$$J\,\mathcal{O}_{W_1} = I(E)^{b^*} J_1 .$$

Y $\qquad$ E

(J_1, b^*) is the pair assigned to π_1.

$$\begin{array}{ccc} S[Z]/u & & R[Z]/Z^b + a_1 Z^{b-1} + \dots + a_b \\ | & & | \\ S & \longrightarrow & R \\ s_i & \longmapsto & (-1)^i b_i \end{array}$$

$(\hookrightarrow a_i\ ?)$

$$(Z-Y_1)(Z-Y_2)\dots(Z-Y_b) =$$

$$u = Z^b - s_1 Z^{b-1} + \dots + (-1)^b s_b$$

$$k[s_1, s_2, \dots, s_b] \subset k[Y_1, \dots, Y_b]$$

$$\begin{array}{c} \operatorname{Spec}\left(S[Z]/u(Z)\right) \\ | \\ \operatorname{Spec}(S) \\ \| \end{array}$$

$R[Z]$ $\quad Z^b + a_1 Z^{b-1} + \dots + a_b$

$Z \to Z - a$ $\qquad$ $\alpha_1 + a$

$\alpha_b + a$

$$\operatorname{Spec}(k[Y_1, \dots, Y_b]) \neq \mathbb{A}^n_k \longrightarrow \mathbb{A}^b_k = \operatorname{Spec}(k[s_1, \dots, s_b])$$

$(\alpha_1, \dots, \alpha_b)$ $\qquad$ x_S

$$\begin{array}{c} K[z]/\bar{u}(z) \\ | \\ K \end{array}$$

Find invariants in $\mathbb{A}_k^n$
by the actions of 2 groups.

$k[T_1, \dots, T_N] \subset k[S_1, \dots, S_b]$

$z^b + a_1 z^{b-1} + \dots + a_b$

1. $V(T_{n_i}(a_1, \dots, a_b))$ is the radicial locus of T_1.

2. if $T_{n_i}(S_1, \dots, S_b)$ w-h. of degree n_i

$$\boxed{\begin{array}{l} J^* = \left\langle \left(T_{n_i}(a_1, \dots, a_b)\right)^{P/n_i} \right\rangle \\ b^* = \prod n_i \end{array}}, \quad \cancel{P = \prod n_i}$$