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"On hypersurface singularities & multiplicity"

f:  $z^2 + xy^2 \in k[x, y, z] = S$ ,  $\text{char}(k) = 0$ .

$V(f) = X \subset \mathbb{A}_k^3 = \text{Spec}(S)$ .

Find all points of mult. 2.

•  $(R, M)$  loc. reg.,  $\mathfrak{J}$  has order  $b$  if  $\mathfrak{J} = M^b$ ,  $\mathfrak{J} \not\subset M^{b+1}$

•  $p \in V(f) = X$ .  $X$  has mult.  $b$  at the point  $p$  if  $\langle f \rangle$  has order  $b$  at  $S_p$ , or  $p$  is a  $b$ -fold point.

1)  $\mathfrak{J} = \langle f \rangle$ ,  $\mathfrak{J}_1 = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$V(\mathfrak{J}_1) \subset V(f) = X$  points of mult. 2 (or more).

2) i)  $z^2 + az + b$   $a, b \in k[x, y]$

$\frac{\partial f}{\partial z} = 2z + a$ ,  $a \neq 0$ ,  $\text{char}(k) \neq 2$

$z \in \mathfrak{J}_1$ ,  $V(\mathfrak{J}_1) \subset V(z) = H$  smooth hypersurface

$$2) ii) \{P \in \text{Spec}(S) \mid \text{order of } f \text{ at } Sp \text{ is } \geq 2\} = \\ \{P \in \text{Spec}(S) \mid z \in P \text{ and order of } XY^2 \geq 2 \text{ at } Sp\}$$

$$H = V(z), \quad \langle XY^2 \rangle \subset \mathcal{O}_H$$

$$= \{x \in H \mid \langle XY^2 \rangle \text{ has order } \geq 2 \text{ at } \mathcal{O}_{H,x}\}$$

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2) i)  $F = V(z_1) \subset H$  smooth hyp.

ii)  $F$  is described by  $(\langle XY^2 \rangle, 2)$

$$\langle XY^2 \rangle \subset \mathcal{O}_H.$$

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- Tschirnhausen

- Generic projection

$$\mathbb{A}^3 \xleftarrow[\pi]{\text{blow up } x_0} W$$

$$X \quad X_1 \text{ s.t.}$$

$$H \quad H_1 \text{ s.t.}$$

$$H \xleftarrow[x_0]{H_1} \text{bl. up at } x_0$$

$$F \ni x_0$$

$$\langle XY^2 \rangle_{H_1} = \mathcal{I}(E)^2 \mathcal{I}_1$$

claim: i) the set of  $k$ -fold pts. in  $X_1 \subset H_1$

ii) the set of  $k$ -fold pts. is described by  $(\mathcal{I}_1, 2)$

$$k[X, Y, Z] \hookrightarrow k\left[\frac{X}{Y}, Y, \frac{Z}{Y}\right]$$

$$Z^2 + XY^2 \quad \left(\frac{Z}{Y}\right)^2 + \left(\frac{X}{Y}\right) Y^2$$

$$\left(\frac{Z}{Y}\right)^2 + a\left(\frac{Z}{Y}\right) + b; \quad a, b \in k\left[\frac{X}{Y}, Y\right]$$

~~$V\left(\frac{Z}{Y}\right) \in$  set of points of mult. 2.~~

2i)  $V\left(\frac{Z}{Y}\right) = H_1^{(1)} \supset$  set of pts. of mult. 2

2ii) In  $V\left(\frac{Z}{Y}\right) = H_1^{(1)}$  the set of pts. of mult. 2 is defined

$$\mathcal{O}_{H_1^{(1)}} \left( \left(\frac{X}{Y} \cdot Y\right), 2 \right)$$

$$H \longleftarrow H_1^{(1)}$$

$$xy^2 \quad y^2 \left(\frac{x}{y} y\right)$$

$$k[x, y] \longrightarrow k\left[\frac{x}{y}, y\right]$$

$$f \in k[x_1, \dots, x_n] \quad V(f) = X \subset \mathbb{A}_k^n$$

$$J = \langle f \rangle \quad J^{(b-1)} = \left\langle f, \frac{\partial f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \mid \sum_i \alpha_i \leq b-1 \right\rangle$$

$$z^b + \alpha_1 z^{b-1} + \dots + \alpha_b, \quad z_1 = z + \frac{1}{b} \alpha_1 \in k[z_1, x_2, \dots, x_n]$$

$$z_1^b + \alpha_2' z_1^{b-2} + \dots + \alpha_b' \quad \alpha_i \in k[x_2, \dots, x_n]$$

$$z_1 \in J^{(b-1)}$$

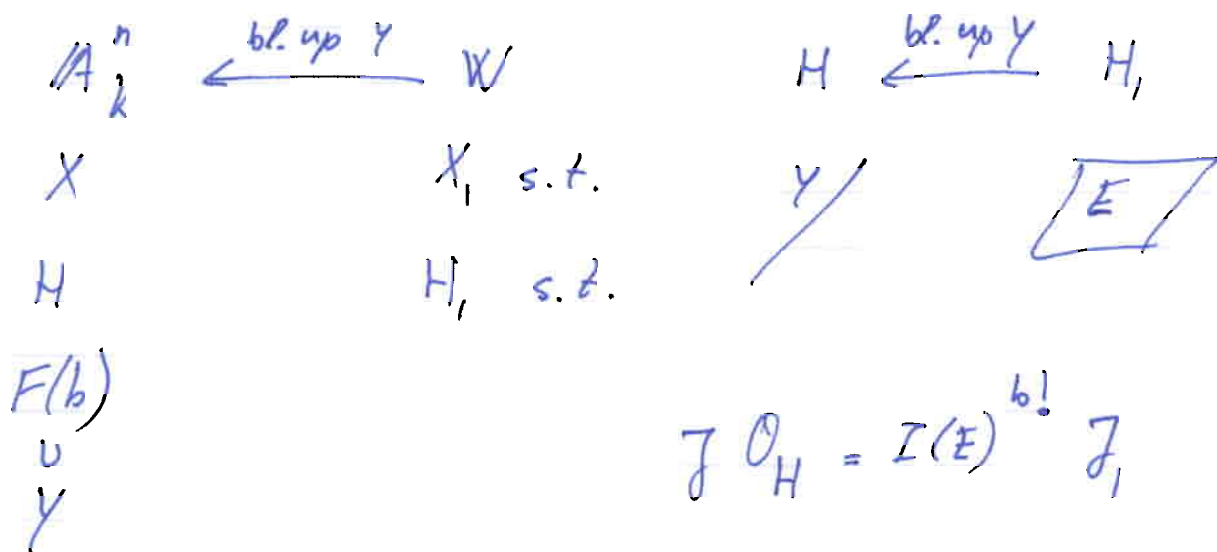
$$F(b) \subset V(z_1) = H \text{ smooth hyp.}$$

b-fold pts.

$$F(b) = \left\{ x \in H \mid \begin{array}{l} \mathcal{O}_{H,x} \\ \text{order of} \\ \alpha_2' \geq 2 \\ \alpha_3' \geq 3 \\ \vdots \\ \alpha_b' \geq b \end{array} \right\}$$

$$J = \left\langle \binom{b!}{\alpha_2'} \frac{b!}{2}, \binom{b!}{\alpha_3'} \frac{b!}{3}, \dots, \binom{b!}{\alpha_b'} \frac{b!}{b} \right\rangle$$

- i)  $F(b) \subset H$   
 ii) described by  $(\mathbb{Z}, b!)$ .



- P1)  $b$ -fold pts. of  $X_1 \subset H$   
 P2)  $b$ -fold pts. of  $X_1$  described  $(\mathbb{Z}_1, b!)$

$X \subset \mathbb{A}_k^n$  hyp. surface.  $x_0 \in X$  closed  $b$ -fold pt.

$\mathcal{O}_{\mathbb{A}_k^n, x_0}$                        $\widehat{\mathcal{O}}_{\mathbb{A}_k^n, x_0}$

Weierstrass:  $f = z^b + a_1 z^{b-1} + \dots + a_b \in R[z]$   
 $R = k[x_2, \dots, x_n]$

$R$  reg.  $B = R[z] / \langle z^b + \alpha_1 z^{b-1} + \dots + \alpha_b \rangle$

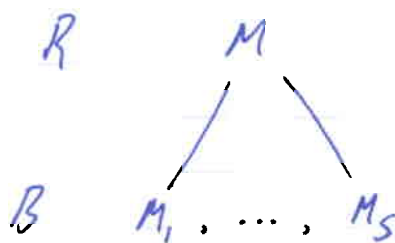
$\alpha_i \in R$

$\hat{X} = \text{Spec} (R[z] / \langle z^b + \alpha_1 z^{b-1} + \dots + \alpha_b \rangle)$

$\downarrow$   
 $\text{Spec}(R)$

$B$   
 $\uparrow$   
 $R$  reg.

Zariski:



$K = R_M / M$

$K_i = B_{M_i} / M_i B_{M_i}$

$b = \sum_{i=1}^s \dim_K (K_i) \cdot e_{M B_{M_i}} (B_{M_i})$

Proposition 1

$\hat{X} = \text{Spec}(B)$

$F(b) \subset \hat{X}$   $b$ -fold pts.

$\pi \downarrow$   
 $\downarrow$   
 $\mathbb{W}$   $\text{Spec}(R)$

$\downarrow \pi$   
 $\pi(F(b)) \subset \mathbb{W}$

Given  $x \in \pi(F(b))$ ,  $\pi^{-1}(x) = \{y\}$  and the residue fields are equal.

$$e_{M_i, B_{M_i}}(B_{M_i}) \leq e(B_{M_i}) = b \quad s=1 \text{ and } K_1 = K.$$

Proposition 2  $F(b) \hat{X} = \text{Spec}(B)$

$$\begin{array}{c} \pi \downarrow \\ \pi(F(b)) \subset W \subset \text{Spec}(R) \end{array}$$

1) if  $Y \subset \pi(F(b)) \subset W$  is smooth & closed  
 $\exists \tilde{Y} \subset F(b)$  mapping to  $Y$  and  $\tilde{Y} \xrightarrow{\sim} Y$

$$\begin{array}{ccc} 2) \tilde{X} & \xleftarrow{\text{bl. up } \tilde{Y}} & X_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ W & \xleftarrow{\text{bl. up } Y} & W_1 \end{array}$$

$\pi_1$  is also a simple cover of degree  $b$ .

Thm. ( $\text{char}(k) = 0$ ) To each simple cover of deg.  $b$   $\xrightarrow{\text{assign}}$  a pair

$$\begin{array}{c} \hat{X} \\ \pi \downarrow \\ W = \text{Spec}(R) \end{array}$$

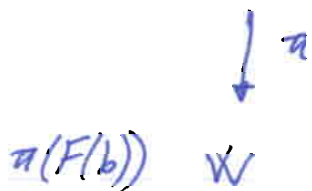
$$(\mathcal{I}, b^*), \quad \mathcal{I} \subset \mathcal{O}_W.$$

P1)  $\pi(F(b))$  described by  $(\mathcal{I}, b^*)$

P2)

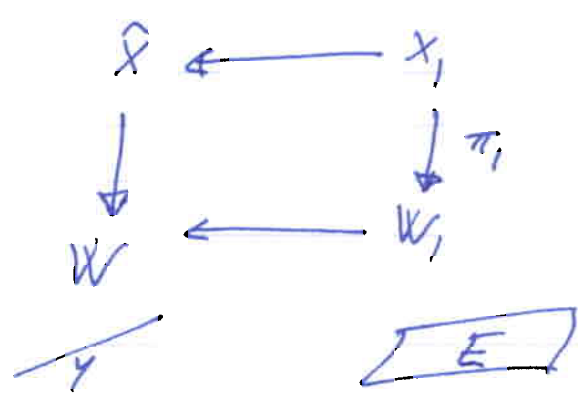


P2)  $F(b) \subset \tilde{X}$



$$\mathcal{I}_{W_1}^0 = I(E)^{b^*} \mathcal{I}_1$$

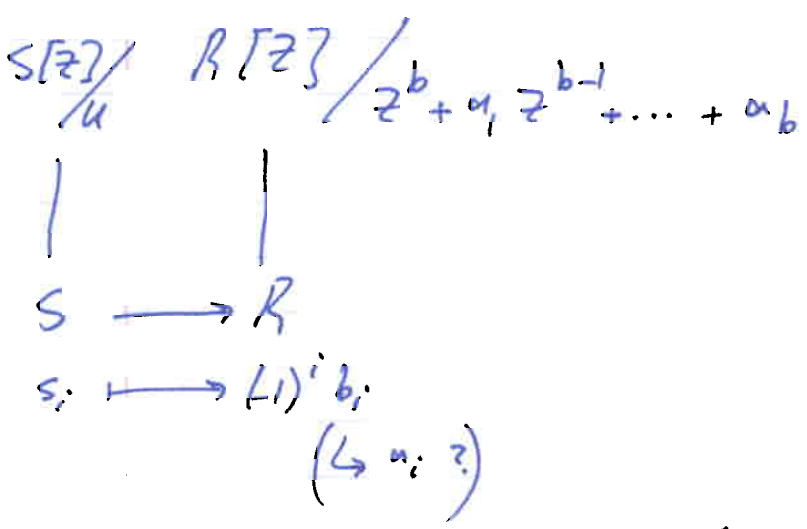
$$Y \subset \pi(F(b))$$



as in Prop. 2

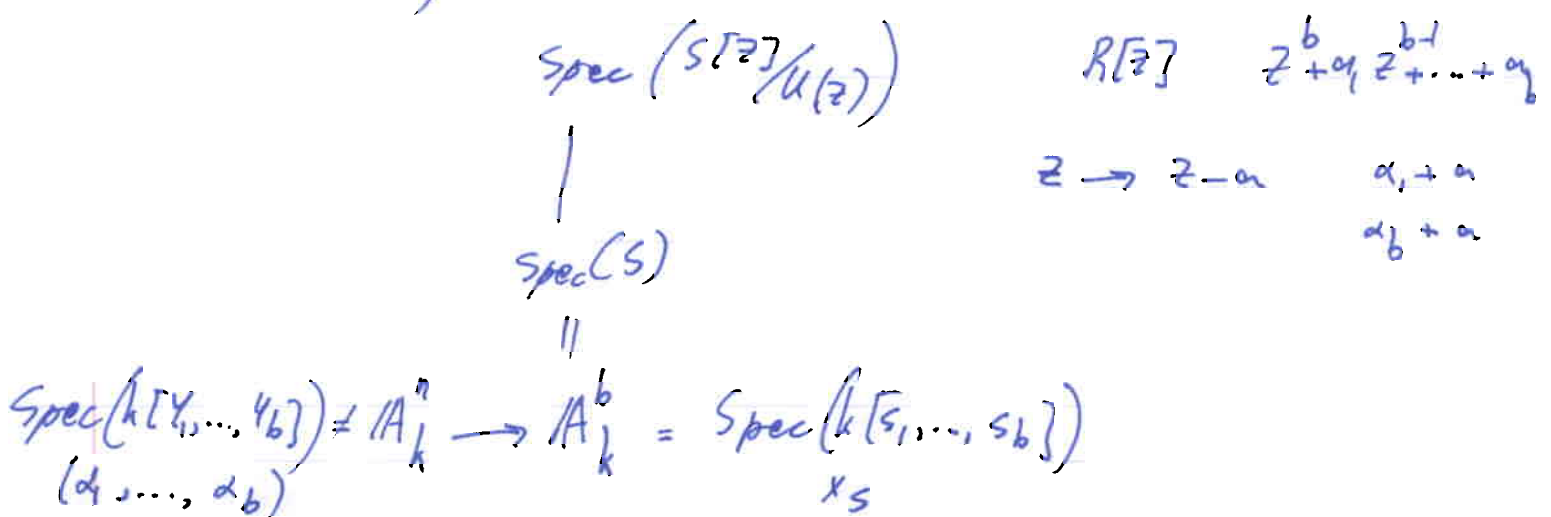
$(\mathcal{I}_1, b^*)$  is the pair assigned to  $\pi_1$ .

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$$(z - y_1)(z - y_2) \dots (z - y_b) = z^b - s_1 z^{b-1} + \dots + (-1)^b s_b$$

$$k[s_1, s_2, \dots, s_b] \subset k[y_1, \dots, y_b]$$





$$K[z]/\bar{u}(z)$$



Find invariants in  $A_k^n$   
by the actions of 2 groups.

$$k[T_1, \dots, T_N] \subset k[s_1, \dots, s_b]$$

$$z^b + a_1 z^{b-1} + \dots + a_b$$

1.  $V(T_{n_i}(a_1, \dots, a_b))$  is the radical locus of  $T_i$ .

2. if  $T_{n_i}(s_1, \dots, s_b)$  w-h. of degree  $n_i$

$$\begin{array}{l} \mathcal{I}^* = \langle (T_{n_i}(a_1, \dots, a_b))^{P/n_i} \rangle \\ b^* = \prod n_i \end{array}$$

$$P = \prod n_i$$

