Eero K. Hyry Dec. 2, '02

"Adjoint-type modules and applications"

joint work with Karen E. Smith

Let X be a smooth projective variety $/\mathbb{C}$,

$\mathcal{L}$ an ample invertible sheaf on X.

When is $\Gamma(X, \mathcal{L}^n) \neq 0$?

Conjecture (Kawamata) $\mathcal{L}$ ample, $\mathcal{L} \otimes \omega_X^{-1}$ ample

then $\Gamma(X, \mathcal{L}) \neq 0$.

(Riemann-Roch $\Rightarrow$ curves, Kawamata $\Rightarrow$ surfaces)

Algebraic formulation

$$S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \qquad (\text{"section ring"})$$

normal domain, $d = \dim S$

Fix $N \gg 0$ $\xRightarrow{\text{(ampleness)}}$ $\exists$ s.o.p. $(x_1, \dots, x_d)$ of deg. N.

ω_S = graded canonical module of S.

$$S_1 = 0 \overset{\text{(Serre duality)}}{\iff} \left[H^d_{S^+}(\omega_S)\right]_{-1} = 0$$

$$\iff [\omega_S]_{Nd-1} = (x_1, \dots, x_d)\,[\omega_S]_{Nd-1-N}$$

$$\iff [\omega_S]_{Nd-1} \subset \bigcap_{\substack{x_1,\dots,x_d \text{ s.o.p.} \\ \text{of deg } N}} (x_1, \dots, x_d)\,\omega_S$$

set $I = S_{\geq N}$. An ideal $J \subset I$ is a reduction of I if $I^{r+1} = J I^r \quad \forall\, r \gg 0$.

$\{$minimal homog. reductions of $I\} = \{$s.o.p. of degree $N\}$

Let (A, m) be a local ring, $d = \dim A$, $I \subset A$ an ideal.

<u>core of I</u>

$$\operatorname{core}(I) = \bigcap_{\substack{J \subset I \\ \text{reduction}}} J \qquad \text{(Rees – Sally 1988)}$$

[recall notion of adjoint of an ideal]

Suppose A is Gorenstein domain (A is essentially of finite type $/\mathbb{C}$).

Let $Z \to \mathrm{Spec}(A)$ be a resolution of singularities s.t. $I\mathcal{O}_Z$ is invertible.

adjoint of I

$$\mathrm{adj}(I) = \Gamma(Z, I\omega_Z) \hookrightarrow \Gamma(Z, \omega_Z) \hookrightarrow \omega_A = A.$$

Thm. (Huneke – Swanson) Suppose A regular, $d = 2$, $I = \overline{I}$ m-primary. Then $\mathrm{core}(I) = \mathrm{adj}(I^2)$

Rmk. In this case Rees algebra $A[It]$ has rational singularities, in particular Cohen-Macaulay.

A any ring, $I \subset A$ an ideal.

core-module of I $\quad \mathrm{core}(I\omega_A) = \bigcap_{\substack{J \subset I \\ \text{reduction}}} J\omega_A$

adjoint-module $\quad \mathrm{adj}(I\omega_A) = \Gamma(Z, I\omega_Z) \hookrightarrow \omega_A$

Let S be a section ring as before, $I = S_{\geq N}$. Then $\operatorname{adj}(I\omega_S) = [\omega_S]_{\geq N+1}$.

<u>Thm. 0</u> $S_1 \neq 0 \iff \operatorname{gradedcore}(I\omega_S) = \operatorname{adj}(I^d\omega_S)$

(Here $\operatorname{gradedcore}(I\omega_S) = \bigcap\limits_{\substack{J \subset I \\ \text{homog. reduction}}} J\omega_S$)

<u>Thm. 1</u> Let (A, m) be a local ring, essentially of finite type $/\mathbb{C}$, and let I be an m-primary ideal s.t. $(A[It])^+$ (irrelevant ideal of the Rees ring) is CM as an $A[It]$-module.

If $\operatorname{Proj}(A[It])$ has rational singularities, then $\operatorname{core}(I\omega_A) = \operatorname{adj}(I^d\omega_A)$.

<u>Rmk.</u> If S is section ring with $\mathcal{I}$, $\mathcal{I} \otimes \omega_X^{-1}$ ample $\Rightarrow S[It]^+$ CM.

About the proof of Thm. 1.

<u>Main Tool</u> "adjoint type modules"

$$\Omega_n = \Gamma(Y, I^n \omega_Y) \qquad (n \in \mathbb{Z})$$

where $Y = \mathrm{Proj}\, A[It]$ (A any local ring, $I \subset A$ ideal)

<u>Properties</u>

1) $\mathrm{adj}(I^n \omega_A) \subset \Omega_n$ and equality holds if Y has rational singularities.

2) $$\cdots \supset \Omega_{-1} \supset \Omega_0 \supset \Omega_1 \supset \cdots$$
$$\Omega_0 \xrightarrow{\text{trace}} \omega_A$$

3) $I\,\Omega_n \subset \Omega_{n+1}$, $\Omega_n = \mathrm{Hom}(I, \Omega_{n+1})$

4) A is CM, I is equimultiple of height h (i.e. ∃ min'l reduction gen. by h elements)

$$\Longrightarrow \quad \Omega_n = J^{n+r-h+1}\omega_A : I^r \qquad (r \gg 0)$$

5) $\omega_{A[It]} = \bigoplus_{n\geq 1} \Omega_n$

I $\mathfrak{m}$-primary, $\mathcal{G} = \bigoplus_{n\geq 0} I^n/I^{n+1}$

$$\Longrightarrow \bigoplus_{n\geq 1} \Omega_{n-1}/\Omega_n \subset \omega_{\mathcal{G}}$$

"Proof of Thm. 1"

We actually showed that $\operatorname{core}(I\omega_A) = \Omega_d$

"$\supset$" Briançon - Skoda for adjoints

$$\Omega_d = \operatorname{adj}(I^d \omega_A) \subset J\omega_A \qquad \forall\, J \subset I \text{ reduction}$$

"$\subset$" one has

$$\bigcap_{\substack{(x_1,\dots,x_d)\subset I \\ \text{reduction}}} (x_1,\dots,x_d)\,\omega_A \subset \Omega_d$$

$$\Uparrow$$

$$\bigcap_{x_1,\dots,x_d} \left[(x_1^*,\dots,x_d^*)\,\omega_{\mathcal{G}}\right]_d = 0 \qquad \longleftarrow \text{induction on } d$$

Results about the core

Let $I \subset A$ be equimultiple of height $h > 0$.

Thm. Let A be regular of essentially finite type $/ \mathbb{C}$.

Suppose that $A[It]$ is CM and normal.

Then $A[It]$ has rational singularities $\Longleftrightarrow$

$\operatorname{core}(I) = \operatorname{adj}(I^{\cancel{d}\,h})$.

If this is the case, then $\operatorname{core}(I) = I \operatorname{adj}(I^{n-1})$,

$\operatorname{adj}(I^{n-1}) = \operatorname{core}(I) : I$.

Thm. Suppose A, $A[It]$ are CM, $\operatorname{char}(A) = 0$.

Then $\operatorname{core}(I) = \Omega_d :_A \omega_A = J^{r+1} : I^r \quad \forall J \subset I$ min'l reduction $(r \gg 0)$

conjectured/proven by Corso-Polini-Ulrich

(If $d = 1$, we can omit $A[It]$ CM.)

Thm. Let S be a standard graded ring over a field. If S is normal and CM, then $\operatorname{core}(\mathfrak{m}^N) = \operatorname{gradedcore}(\mathfrak{m}^N) = \mathfrak{m}^{N+a+1}$ where $\mathfrak{m}$ = the homog. maximal ideal of S, $a = a(S)$.