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"Ramification and resolution in transcendental extensions"

Algebraic Geometry $\leadsto$ local algebra

X = projective variety $/k$ $\qquad K = k(X)$

local rings $\{\mathcal{O}_{X,p}\}$ $\qquad \mathcal{O}_{X,p} \subset K$

$k \subset V \subset K$ a valuation domain

$\exists!\ p \in X$ s.t. $\mathcal{O}_{X,p} \subset V$

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ex W = surface

$$W \longleftarrow W_1 \longleftarrow W_2 \longleftarrow \cdots \longleftarrow W_n \longleftarrow \cdots$$

normalize, and blow up the finitely many sings.

is this finite?

If it is <u>not</u> finite (all W_n are singular)

$$p \longleftarrow p_1 \longleftarrow p_2 \longleftarrow$$

$R_i = \mathcal{O}_{W_i, p_i}$

$R \to R_1 \to R_2 \to \cdots$ <u>not</u> regular

$\bigcup_{i=1}^{\infty} R_i = V$ is a valuation ring

But: perhaps V is <u>not</u> Noetherian.

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k a field, K alg. function field $/k$

V val. domain of K/k

$v : K^* \longrightarrow \Gamma_v$ = value group

$V = \{ f \in K \mid v(f) \geq 0 \}$

<u>ex</u> S = "formal" power series

$h = \sum \alpha_i t^{\sigma_i}$ $\alpha_i \in k$ $\sigma_i \in \mathbb{R}$

$\sigma_i < \sigma_j$ for $i < j$

$v(h) = \sigma_0$

$g_1, \dots, g_n \in S$ $K = k(x_1, \dots, x_n)$

$K \hookrightarrow S$

$f \longmapsto f(g_1, \dots, g_n)$

$$\nu(f) = \bar{\nu}\left(f(z_1, \dots, z_n)\right) \in \mathbb{R}$$

e.g. $\Gamma_\nu = \mathbb{Q}$

A local ring R of K is <u>algebraic</u> if R is essentially of finite type$/k$, $Q(R) = K$

monoidal transform along V

$R \subset V \subset K$, $P \subset R$ regular prime.

alg local ring

($\Rightarrow R/p$ is a RLR.)

$\exists\, x \in p$, $\nu(x) = \min\{\nu(f) \mid f \in p\}$

$$\begin{array}{c} R \longrightarrow R\left[\frac{p}{x}\right]_{m_V \cap R[\frac{p}{x}]} \longrightarrow V \\ \quad \| \\ \quad R_1 \end{array}$$

R regular $\Rightarrow R_1$ regular.

<u>ex</u> $R/m_R = R_1/m_{R_1} \subset R$

$\Rightarrow$ $\exists$ reg pars $(x_1, \dots, x_n)$ in R,

$(\bar{x}_1, \dots, \bar{x}_n)$ in R_1, $P = (x_1, \dots, x_r)$

$$x_i = \begin{cases} \bar{x}_i \cdot \bar{x}_r & i < r \\ \bar{x}_i & i \geq r \end{cases}$$

Local uniformization (Zariski)

char$(k) = 0$

$\exists$ regular algebraic local ring R of K s.t. $\underline{\underline{R \subset V}}$

$\exists$ RLRs R_i of K $\varinjlim R_i = V$

$K \to K^*$ possibly transcendental extension

of alg. func. fields over k.

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local unit in dim ≤ 3 $\Rightarrow$ resolution of sings

in dim ≤ 3

$$\begin{tikzcd} K \arrow[r] & K^* \\ V = V^* \cap K \arrow[u] \arrow[r] & V^* \arrow[u] \end{tikzcd}$$

$$\nu^* : K^* \twoheadrightarrow \Gamma^*$$

$$\nu : \nu^*|_K \twoheadrightarrow \Gamma$$

$$\Gamma < \Gamma^*$$

<u>Theorem</u> (local monomialization) $\quad char(k) = 0$

Given
$$\begin{tikzcd} K \arrow[r] & K^* \\ V \arrow[u] \arrow[r] & V^* \arrow[u] \\ \bar{R} \arrow[u] \arrow[r] & \bar{S} \arrow[u] \end{tikzcd} \text{ alg. local rings} \quad \Longrightarrow \quad \exists \begin{tikzcd} V \arrow[r] & V^+ \\ R_0 \arrow[u] \arrow[r] & S \arrow[u] \\ \bar{R} \arrow[u, "\alpha"] \arrow[r] & \bar{S} \arrow[u, "\beta"'] \end{tikzcd} \qquad (*)$$

R_0, S regular; α, β are products of monoidal transforms

$\exists$ reg. pars $(x_1, \dots, x_m)$ in R_0, $(y_1, \dots, y_n)$ in S,

$\delta_i \in S$, $\qquad A = (a_{ij})$

$$x_i = \prod_{j=1}^{n} y_j^{a_{ij}} \delta_i \qquad (1 \le i \le m)$$

$$\boxed{\operatorname{rank}(A) = m \le n}$$

case K^*/K finite : Astérisque '99

general case : preprint

rank $(A) = m$ (and char $(k) = 0$)

$\Longrightarrow$ $\exists$ an étale extension $S \longrightarrow \tilde{S}$,

reg. pars $(\bar{y}_1, \ldots, \bar{y}_n)$ in $\tilde{S}$

$$X_i = \prod_{j=1}^{n} \bar{y}_j^{\,u_{ij}} \quad (1 \le i \le m)$$

If $k = K$ then we recover Zariski's local uniformization theorem.

Corollary Given a system

$$(+) \qquad \begin{aligned} X_1 &= f_1(Y_1, \ldots, Y_n) \\ &\vdots \\ X_m &= f_m(Y_1, \ldots, Y_n) \end{aligned}$$

$\Longrightarrow$ there are finitely many charts which are compositions of monoidal transforms.

(*)

$$X_i = \phi_i(\overline{X}_1, \dots, \overline{X}_m) \qquad (1 \le i \le m)$$
$$Y_i = \psi_i(\overline{Y}_1, \dots, \overline{Y}_n) \qquad (1 \le i \le n)$$

$$\overline{X}_1 = \overline{Y}_1^{a_{11}} \cdots \overline{Y}_n^{a_{1n}}$$
$$\vdots$$
$$\overline{X}_m = \overline{Y}_1^{a_{m1}} \cdots \overline{Y}_n^{a_{mn}}$$

All solutions to (+) are transformed into one of these charts.

⊛ a composition of (1) a change of variables

$$(2) \quad \begin{cases} X_1 = X_1(1)\, X_2(1) \\ X_i = X_i(1) & i > 1 \end{cases}$$

<u>Corollary</u> (weak local simultaneous resolution)

K^*/K is finite, $\mathrm{char}(k) = 0$,

then $\exists$

$$(**) \quad \begin{tikzcd} V \arrow[r] & V^* \\ R \arrow[u] \arrow[dr] & \\ R_0 \arrow[u] \arrow[r] & S \arrow[uu] \\ \overline{R} \arrow[u] \arrow[r] & \overline{S} \arrow[u] \end{tikzcd}$$

a factorization of $(*)$ s.t. R has toric singularities.

$S =$ localization of the integral closure of R in K^* (S is regular)

<u>Thm.</u> (—) $\exists$ a proper morphism $f: W \to T$ of projective surfaces s.t.

$$\nexists \quad \begin{tikzcd} W_1 \arrow[r, "f_1"] \arrow[d, "\alpha"'] & T_1 \arrow[d, "\beta"] \\ W \arrow[r, "f"] & T \end{tikzcd}$$

α, β proper birational
f_1 finite
W_1 non-singular

counterexample to "weak global simultaneous resolution"

<u>Theorem</u> (Piltant, —) $\exists$ ALR R_1 of K s.t. ($R_1 = R_1(V, V^*)$)

if $R_1 \subset R_0$ and K^*/K is finite and:

$$\begin{tikzcd} V \arrow[r] & V^* \\ R \arrow[u] \arrow[r, "\text{"finite"}"] & S \arrow[u] \\ R_0 \arrow[u] \arrow[ur, "x_i = \prod y_j^{a_{ij}} s_i"'] & \end{tikzcd} \quad (1 \le i \le m)$$

is as in $(**)$

then: $(A = (a_{ij}))$

1. $\mathbb{Z}^n / A^t \mathbb{Z}^n \xrightarrow{\sim} \Gamma^* / \Gamma$

$(b_1, \dots, b_n) \longmapsto b_1 \nu^*(y_1) + \dots + b_n \nu^*(y_n)$

is an isomorphism.

2. $\bar{k}$ = alg. closure of $k(V^*)$

$$\hat{R} \otimes_{k(R)} \bar{k} \cong \left(\hat{S} \otimes_{k(S)} \bar{k}\right)^{\Gamma^*/\Gamma}$$

3. $e = [\Gamma^* / \Gamma] = |\det(A)|$

<u>Theorem</u> (Piltant, –)

$$\begin{array}{ccc} K & \longrightarrow & K^* \\ \uparrow & & \uparrow \\ V & \longrightarrow & V^* \end{array}$$ finite, $\operatorname{char}(k) = 0$

$\Rightarrow$ $\exists$ normal alg. local rings R_i of K,

regular " " " S_i of K^*

$V = \varinjlim R_i$, $V^* = \varinjlim S_i$

set $A = \varinjlim \hat{R}_i$

$B = \varinjlim \hat{S}_i$

$Q(B)$ is Galois over $Q(A)$ with Galois group Γ^*/Γ

$A = B^{\Gamma^*/\Gamma}$

In general (Ghezzi, _):

B is not a valuation ring

But it is almost...

Theorem (Piltant, _)

$\mathrm{char}(k) = p > 0$, $k = \bar{k}$, $K \to K^*$ 2 dim alg. fun fields

Local monomialization holds if ν^* is a val. of K^* which has no defect over K.