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"p- standard systems of parameters"

$(A, \mathfrak{m})$ Noetherian local ring

M f.g. A-module of dim $d > 0$

$\mathfrak{a}(M) = \prod_{i<d} \operatorname{ann} H^i_{\mathfrak{m}}(M)$

Definition $x_1, \dots, x_d$: p-standard s.o.p. of type $s<d$

if $x_{s+1}, \dots, x_d \in \mathfrak{a}(M)$

$x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M)$ $(i \leq s)$

Cuong 1991

If $\exists$ dualizing complex then $\exists\, x_1 \cdots x_d$:

p-std. s.o.p. of type $d-1$

(1) $\ell(M/(x_1^{n_1}, \dots, x_d^{n_d})M)$ is a polynomial of $n_1 \cdots n_d$

(2) $\forall\, \Lambda \subset \{1 \dots d\}$ $n_1 \cdots n_d$, $i \notin \Lambda$

$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i} = (\text{———}) M : x_i$

i.e. unconditioned $\mathfrak{p}$-seq.

Cuong 1995

$x_1 \dots x_d$ $\mathfrak{p}$-std. s.o.p. $\Rightarrow$

$\forall\, n_1 \dots n_d$, i, j

$(x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} x_j^{n_j} = (x_1^{n_1} \dots x_{i-1}^{n_{i-1}})M : x_j^{n_j}$

Thm. 1. $x_1 \dots x_d$: $\mathfrak{p}$-std. s.o.p. of types
$(\#s)$ $\quad \Lambda \subset \{1 \dots d\}$

$n_1 \dots n_d > 0 \qquad i, j \notin \Lambda$

$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i} x_j^{n_j} = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_j^{n_j}$

if $i < j$ or $j > s$.

Prop. 2 (Schenzel 1979)

$x_1 \dots x_d$: s.o.p. of M

$(x_1 \dots x_{i-1}) M : x_i \subset (\text{——}) M : \mathfrak{a}(M)$

Pf. of Thm. 1

If $j > s$ then both sides $= (\text{——}) M : \mathfrak{a}(M)$

Assume $j \leq s$ $\quad p = \#\{\lambda \in \Lambda \mid \lambda > j,\ n_\lambda > 1\}$

If $p = 0$, $q = \# \{ \lambda \in \Lambda \mid \lambda > j \}$

If $q = d - j$ i.e. $j+1 \ldots\ldots d \in \Lambda$

both side $= (\sim\!\!\sim) M : \alpha\, (M/(x_{j+1}, \ldots, x_d) M)$

$q < d - j$ $k \notin \Lambda$ s.t. $k+1 \ldots d \in \Lambda$

Let $\alpha \in (\sim\!\!\sim) M : x_i^{n_i} x_j^{n_j}$

$$\alpha \in [(\sim\!\!\sim) M + x_k M] : x_i^{n_i} x_j^{n_j}$$

$$= [(\sim\!\!\sim) M + x_k M] : x_j^{n_j}$$

$$x_j^{n_j} \alpha \in [(\sim\!\!\sim) M + x_k] \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i}]$$

$$= (\sim\!\!\sim) M + x_k M \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i}]$$

$$= (\sim\!\!\sim) M + x_k [(\sim\!\!\sim) M : x_i^{n_i} x_k]$$

$$\subseteq (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M$$

Since $k+1 \ldots d \in \Lambda$

$$(\sim\!\!\sim) M : x_i^{n_i} x_k = (\sim\!\!\sim) M : x_k$$

Assume $p > 0$

$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i} \subset (\sim) M : x_j^{n_j}$$

$$(\sim) M : x_j^{n_i+n_j} \subset (\sim) M : x_j^{n_j}$$

$$(\sim) M : x_j^{n_j} \subset (\sim) M : x_i^{n_i} x_j^{n_j}$$

$$a \in (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i}$$

$$\exists \mu \in \Lambda \quad \text{s.t.} \quad \mu > j, \quad n_\mu > 1$$

$$\Lambda' = \Lambda \setminus \{\mu\}$$

$$x_i^{n_i} a = b + x_\mu^{n_\mu} c, \qquad b \in (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda')$$

$$c \in [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda') M + x_i^{n_i} M] : x_\mu^{n_\mu}$$

$$= [\sim] : x_\mu$$

$$x_\mu c = x_i^{n_i} a' + b', \qquad b' \in (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda') M$$

$$a' \in [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda') M + x_\mu M] : x_i^{n_i}$$

$$a - x_\mu^{n_\mu - 1} a' \in (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda') M : x_i^{n_i}$$

Thus
$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M = x_\mu^{n_\mu - 1}\left[\left((x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M + x_\mu M\right) : x_i^{n_i}\right]$$
$$+ (x_\lambda^{n_\lambda} \mid \lambda \in \hat{\Lambda}) M : x_i^{n_i}$$
$$\subseteq (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_j^{n_j}$$

Prop. 3 Let $x_1 \dots x_d$ be a seq. satisfying (#s), $\Lambda \subset \{1 \dots d\}$, $n_1, \dots, n_d > 0$, $i \notin \Lambda$.
$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) M : x_i^{n_i} = \sum_{\Lambda' \subset \Lambda} \left(\prod_{\lambda \in \Lambda'} x_\lambda^{n_\lambda - 1} \cdot (x_\lambda \mid \lambda \in \Lambda') M : x_i \right)$$

Prop. 4 $x_1, \dots, x_d$ satisfying (#s)

$\forall\, n_1, \dots, n_d, m_1, \dots, m_d$
$$(x_1^{n_1 + m_1}, \dots, x_d^{n_d + m_d}) M : x_1^{m_1} \dots x_d^{n_d}$$
$$= \sum_{i=1}^{d} (x_1^{n_1} \dots \hat{x}_i \dots x_d^{n_d}) M : x_i$$
$$+ (x_1^{n_1} \dots x_d^{n_d}) M .$$

Prop. 5 $x_1, \dots, x_d$ satisfies (#s)
$$H^i_{(x_1 \dots x_d)}(M) = \varinjlim_n \frac{(x_1^n, \dots, x_i^n) M : x_{i+1}}{(x_1^n, \dots, x_i^n) M} \qquad (i < d)$$

Cor. 6 $x_1 \dots x_d$ satisfies (#S)

$n_1 > 0,\ n_2 > 1,\ \dots,\ n_s > s-1;\ n_{s+1}, \dots, n_d > d$

$\Rightarrow x_1^{n_1} \dots x_d^{n_d}$ is a p-std. s.o.p. for M of type s.

Let K_M be the canonical module of M,

i.e. $K_M \otimes \hat{A} = \mathrm{Hom}(H_m^d(M), E(A/m))$

Thm. 7 Assume $d > 2$, $\exists K_A$, $x_1, \dots, x_d$ p-std. s.o.p. for M.

For $t \geq 2$ TFAE:

(1) depth $K_M \geq t$

(2) $x_1, \dots, x_t$ is K_M-seq.

(3) $\forall\ i < t-1$

$$(x_1, \dots, \hat{x}_i, \dots x_d)M : x_i \subset \sum_{j=t-1}^{d} (x_1 \dots \hat{x}_i, \dots \hat{x}_j \dots x_d)M : x_j$$

Sketch of proof:

$$E = H_m^d(M) = \varinjlim_n M/(x_1^n, \dots, x_d^n)M$$

$\varphi_n : M/(x_1^n, \dots, x_d^n)M \longrightarrow E$ canonical map

Claim TFAE

(1) $x_{i+1}\,[0 :_E (x_1, \dots, x_i)] = 0 :_E (x_1 \dots x_i)$ $(i \le t-2)$

$\exists y \in (x_t, \dots, x_d)A$ s.t.

$y\,[0 : (x_1 \dots x_{t-1})] = 0 :_E (x_1 \dots x_{t-1})$

(2) $x_{i+1}\,[0 :_E (x_1 \dots x_i)] = 0 :_E (x_1 \dots x_i)$ $(i \le t-1)$

(3) Same as Thm. 7.

(2) of thm. 7 $\Longleftrightarrow$

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
K_M^i \arrow[r] \arrow[d, "x_{i+1}"] & K_M \arrow[r] \arrow[d, "x_{i+1}"] & K_M/(x_1 \dots x_i)K_M \arrow[r] \arrow[d, "x_{i+1}"] & 0 \\
K_M^i \arrow[r, "(x_1 \dots x_i)"'] & K_M \arrow[r] & K_M/(x_1 \dots x_i)K_M \arrow[r] & 0
\end{tikzcd}$$

(ex)

(2) of claim $\Longleftrightarrow$

$$
\begin{tikzcd}
 & & 0 & \\
E^{i} & E \arrow[l] & 0 :_E (x_1 \cdots x_i) \arrow[l] \arrow[u] & 0 \arrow[l] \\
E^{i} \arrow[u] & E \arrow[l, "\begin{pmatrix} x_1 \\ \vdots \\ x_i \end{pmatrix}"'] \arrow[u] & 0 :_E (x_1 \cdots x_i) \arrow[l] \arrow[u, "x_{i+1}"'] & 0 \arrow[l]
\end{tikzcd}
$$

(ex)

$$0 : (x_1 \cdots x_t) = \bigcup T_{n+1}\left(\overline{x_1^n \cdots x_t^n M}\right)$$

$$+ \sum_{i=1}^{t-1} T_2\left(x_1 \cdots \hat{x}_i \cdots x_d \left[\overline{(x_1 \cdots \hat{x}_i \cdots x_d)M : x_i}\right]\right)$$