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"Jet Schemes ~~and multiplier ideals~~"

↳ "Jet schemes and the log canonical threshold"

(X, Y)

1. X smooth complex variety of dim. n
2. $Y \hookrightarrow X$

Log resolution

$f : X' \longrightarrow X$ proper birational

1. X' smooth
2. $f^{-1}(Y) = \sum_1^r a_i E_i$

$K_{X'/X} = \operatorname{div} \det(J(f)) = \sum_1^r k_i E_i$

E_i's are distinct sm. irred. divisor

$\sum_1^r E_i$ has only simple normal crossings

$\lambda \in \mathbb{R}^+$

$(X, \lambda \cdot Y)$ is log canonical if $k_i + 1 - \lambda a_i \geq 0 \ \forall i$

———"——— is canonical if $k_i + 1 - \lambda a_i \geq 1$

———"——— is terminal if $k_i + 1 - \lambda a_i > 1$

(E_i's are f-exceptional)

log canonical threshold

$$lc(X, Y) = \inf_{a_i \neq 0} \frac{k_i + 1}{a_i}$$

$W \subseteq X$ closed subscheme $\qquad f^{-1}(W) = \sum_1^r w_i E_i$

$$mld(W; X, \lambda Y) = \inf_{f(E_i) \subseteq W} k_i + 1 - \lambda a_i$$

(mld: minimum log discrepancy)

$p \in X$. $I_{Y,p}$ is M_p primary in $\mathcal{O}_{X,p}$, $\dim = n$

$$e(I_{Y,p}) = \text{Samuel multiplicity}$$
$$= \ell\left(\mathcal{O}_{X,p}/f_1, \dots, f_n\right)$$

with $f_1, \dots, f_n$ general elements in $I_{Y,p}$

[Y is 0-dim'l closed subscheme here.]

$$e(I_{Y,p}) \geq \frac{n^n}{lc(X,Y)_p^n}$$

$X = \mathbb{A}^n \qquad I_Y = \langle x_1^{d_1}, \ldots, x_n^{d_n} \rangle$

$$e(I_Y) = \prod_{i=1}^n d_i$$

$$lc(X,Y) = \sum_1^n \frac{1}{d_i}$$

$$\prod_{i=1}^n d_i \geq \frac{n^n}{\left(\sum_i \frac{1}{d_i}\right)^n}$$

$$\frac{1}{n}\sum_{i=1}^n \frac{1}{d_i} \geq \left(\prod \frac{1}{d_i}\right)^{1/n}$$

<u>Thm.</u> (DEM) Let X be a sm. hypersurface of deg. n in $\mathbb{P}^n_{\mathbb{C}}$. Assume that $4 \leq n \leq 12$. Then $\mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}(X)$ is a finite group.

$n=4$ Isckovskikh & Manin

<u>Cor.</u> Let X be as above. Then X is not rational.

$n=5$ Pukhlikov & generic hypersurf. degree n hypersurf. in $\mathbb{P}^n$

$n=6,7$ Cheltsov

$$\mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}(X) = \mathrm{Bir}_{\mathbb{C}}(X) = \mathrm{Aut}_{\mathbb{C}} X$$

X_1 & X_2 two hypersurf. of deg. n

$p^{-1}(B) = E$

$$\begin{tikzcd} & X' \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 \arrow[rr, dashed, "f"] & & X_2 \\ H_1 & & H_2 \end{tikzcd}$$

p, q proper birational morphisms

hyperplane classes

$$p_* q^* H_2 \sim d H_1$$

$$V \subseteq H^0(\mathcal{O}_{X_1}(dH_1))$$

$B =$ base locus of $|V|$

Noether Fano inequality

if $(X_1, \frac{1}{d} B)$ is canonical, then f is an isomorph.

$K_{X'/X_1} - \frac{1}{d} E$ is effective (E is pullback of B.)

Using the relation $p^* dH_1 - E_1 = q^* H_2$

we have

$$K_{X'/X_1} - \frac{1}{d} E = K_{X'/X_2} + \left(\frac{1}{d} - 1\right) p^* H_1$$

C_2 : pullback of a generic complete curve to X'

$$\left(K_{X'/X_1} - \frac{1}{d} E\right) \cdot C_2 \geq 0$$

$$\shortparallel \qquad\qquad\qquad d = 1$$

$$q^* \left(\frac{1}{d} - 1\right) H_1 \cdot C_2$$

Jet schemes

X_k = k-th jet scheme of X

$= \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{Spec}\, \mathbb{C}[t]/t^{k+1}, X\right)$

X_1 : tangent bundle

Arcs space : $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}\, \mathbb{C}[[t]], X)$

similarly for Y_k, Y_∞

$X = \mathbb{A}^n$

$$\mathbb{C}[X_1, \dots X_n] \longrightarrow \mathbb{C}[t]/t^{k+1}$$

$$\begin{matrix} x_1(t) \\ \vdots \\ x_n(t) \end{matrix}$$

$$X_k \sim \mathrm{Spec}\, \mathbb{C}[X_1, \dots, X_n, X_1', \dots, X_n', X_1^{(2)}, \dots, X_n^{(2)}, X_1^{(k)}, \dots, X_n^{(k)}]$$

$Y :\quad f = 0$

$$\begin{tikzcd} \mathbb{C}[X_1 \dots X_n] \arrow[r] \arrow[dr] & \mathbb{C}[t]/t^{k+1} \\ & \mathbb{C}[X_1 \dots X_n]/f \arrow[u] \end{tikzcd}$$

$$Y_k = \mathrm{Spec}\, \mathbb{C}[X_1 \dots X_n, X_1' \dots X_n' \dots X_1^{(k)} \dots X_n^{(k)}] \Big/ f, f', \dots, f^{(k)}$$

$$X_\infty \xrightarrow{\psi_m} X_m \xrightarrow{\pi_m} X$$

$D \subseteq X$ normal hypersurface

<u>Thm. A.1.</u> (Mustata)

D has canonical singularity (= rational singularity)

$\longleftrightarrow$ D_k is irred. for all k.

2. (_ , Mustata, Yasuda)

D has terminal singularity $\longleftrightarrow$

D_k is irred. & normal.

(X, Y) D, $W \subseteq D$, $D \not\subseteq \operatorname{supp} Y$

<u>Thm. 8.</u> (inversion of adjunction)

$\operatorname{mld}(W; X, D + \lambda Y) = \operatorname{mld}(W; D, \lambda Y|_D)$

(Kollar: "$\leq$")

$X_\infty \supseteq \mathcal{A}$

$\mathcal{A}$ is said to be a locally closed cylinder if $\mathcal{A}$ is of the form $\psi_m^{-1}(A)$ for some locally closed subset $A \subseteq X_m$.

Assume that A does not dominate X.

$Z \subseteq X$ closed subscheme.

$\underline{\underline{\mathcal{A} \text{ is module}}}$

$\mathrm{Ord}_{\mathcal{A}} I_Z = \min_{r \in \mathcal{A}} \mathrm{ord}_r I_Z$

$\mathrm{ord}_{\mathcal{A}}$ is a discrete valuation for $\mathbb{C}(X)$

$lc(X, Y) = c \iff$ for every irred. closed cylinder $\mathcal{A}$ in X_∞

$\mathrm{codim}(\mathcal{A}, X_\infty) \geq c \cdot \mathrm{ord}_{\mathcal{A}} I_Y$

(equality for some $\mathcal{A}$)

$mld(W; X, \lambda Y) = \tau \iff \mathrm{codim}(\mathcal{A}, X_\infty) \geq \lambda \, \mathrm{ord}_{\mathcal{A}} I_Y + \tau$

$\mathcal{A}$ closed cylinder in $\Psi_0^{-1}(W)$

Thm. B $\Longrightarrow$ Thm. A

$W = \mathrm{Sing}\, D \qquad Y = \emptyset$

$mld(W; X, D) = mld(W; D, \emptyset)$

- D log canonical $\iff$ $mld(W; D, \emptyset) \geq 0$

$mld(W; X, D) \geq 0 \longleftrightarrow \operatorname{codim}(\mathcal{U}, X_\infty) \geq \operatorname{ord}_{\mathcal{U}} I_D$

$\longleftrightarrow$ $A \subseteq D_m$ $\operatorname{codim}(A) \geq m+1$

$\overset{\shortparallel}{\pi_m^{-1}(W)}$

$f, f', \dots, f^{(m)}$ in X_m

D_m is a local complete intersection

- D has canonical sing.

$mld(W; X, D) \geq 1 \longleftrightarrow$ $A \subseteq D_m$

$\overset{\shortparallel}{\pi_m^{-1}(A)}$

D_m is irreducible $\Longleftarrow$ $\operatorname{codim}(A, X_m) \geq m+2$

- D terminal: $\operatorname{codim}(A, X_m) \geq m+3$

sing $D_m = \pi_m^{-1}(W) \cap D_m \Longrightarrow D_m$ is normal.