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"On the residue theorem for formal schemes"

Residue Thm for formal schemes

1. Intro: curves $\rightsquigarrow$ generalizations
2. Motivation (aesthetic)
3. "concrete" residues
4. Globalization: base change, fundamental class
5. formal underpinning (5 operations)

V projective over k (alg. closed)

Ω_V sheaf of holomorph. diff's

$K = k(V)$ $\quad \Omega_K = \Omega_V \otimes_{\mathcal{O}_V} K$

$\omega \in \Gamma(V, \Omega_K)$, $v \in V$, t parameter

$$\omega = \sum_{n \gg -\infty} (a_n t^n)\, dt, \qquad \mathrm{res}_v(\omega) = a_{-1}$$

Ind't of t, $\frac{1}{2\pi i} \oint \omega$ or algebraically

Res Thm. $\sum_{v \in V} \operatorname{res}_v(\omega) = 0$

$$\sum_{v \in V} \operatorname{res}_v(\omega) = \int_{\partial V} \omega$$

alg. proofs see Chevalley, Serre, ...

reformulate: ω holomorph. at $v \iff a_n = 0$ for $n < 0$

$\implies \operatorname{res}_v(\omega) = 0$

So $\operatorname{res}_v : \Omega_K / \Omega_{V,v} \longrightarrow k$

$$\Omega_K / \Omega_{V,v} = H^1_v(\Omega_V)$$

$$0 \to \Omega_{V,v} \to \Omega_K \longrightarrow \Omega_K / \Omega_{V,v} \longrightarrow 0$$

Globally: $0 \to \Omega_V \to \Omega_K \to \Omega^* \longrightarrow 0$

$$\Omega^*(U) = \bigoplus_{v \in U} \Omega_K / \Omega_{V,v} = \bigoplus_{v \in U} H^1_v(\Omega_V)$$

apply $\Gamma(V, -)$, get exact

$$\Omega_K \longrightarrow \bigoplus_{v \in V} H^1_v(\Omega_V) \longrightarrow H^1(\Omega_V) \longrightarrow 0$$

$$\begin{tikzcd} \Omega_K \arrow[r] & \bigoplus_{v\in V} H^1_v(\Omega_V) \arrow[r] \arrow[dr, "\sum \mathrm{res}_v"'] & H^1(\Omega_V) \arrow[r] \arrow[d, "\exists \int"] & 0 \\ & & k & \end{tikzcd}$$

Res. Thm. $\Longleftrightarrow$ $\exists!\ \int : H^1(V, \Omega_V) \longrightarrow k$

s.t.

$$\begin{tikzcd} H^1_v(\Omega) \arrow[rr] \arrow[dr, "\mathrm{res}_v"'] & & H^1(\Omega) \arrow[dl, "\int"] \\ & k & \end{tikzcd}$$

commutes $\forall\ v$.

Generalized to arbitrary proper k-varieties

(see Asterisque 117)

Hubl - Sastry Amer. J. 1993

for certain maps of Noeth. schemes

" " " " formal "

Steps: 1. (Comm. Alg.) $d = \dim V$

Define $H^d_v(\Omega_V) \longrightarrow k$

2. (Geometry) $\exists!\ \int : H^d(\Omega^d_V) \to k$

yielding a similar comm. diagram as above ...

Concrete residues: (Hopkins, –)

in C.R. Math. Rep. Acad. Sci. Canada 1979

Given $R \xrightarrow{f} S \xrightarrow{i} U$ homom. of rings

U f.g. and projective over R.

$$\begin{tikzcd} S \otimes_R U = V \arrow[r, shift left, "p"] & U \arrow[l, shift left, "\xi"] \\ S \arrow[u, "\nu"] \arrow[ur, "i"] & R \arrow[u, "\mu"'] \arrow[l, "f"] \end{tikzcd}$$

$p\nu = i$

$p\xi = 1$

Take: $P_\bullet \longrightarrow U$ V-proj. resolution $\therefore$ S-projective

Have: $e : \mathrm{Hom}_S(V, U) \longrightarrow \mathrm{Hom}_R(U,U) \xrightarrow{\sim}$

$$\xrightarrow{\sim} U \otimes_R \mathrm{Hom}_R(U, R) \longrightarrow \mathrm{Hom}_R(U,R)$$

$$e(\sigma)[u] = \mathrm{Trace}_{U/R}\left(u' \longrightarrow \sigma(1 \otimes uu')\right)$$

Maps of complexes:

$$\mathrm{Hom}^\bullet_S(P_\bullet, U) \simeq \mathrm{Hom}^\bullet_V(P_\bullet, \mathrm{Hom}_S(V,U)) \xrightarrow{\text{via } e} \mathrm{Hom}^\bullet_V(P_\bullet, \mathrm{Hom}_R(U,R)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \mathrm{Hom}^{\cdot}_R (P_{\cdot} \otimes_V U, R)$$

Pass to homology

$$\mathrm{Ext}^q_S(U,U) \to \mathrm{Hom}_R(\mathrm{Tor}^V_q(U,U), R) \qquad \text{any } q$$

or

$$\mathrm{res}_q : \mathrm{Ext}^q_S(U,U) \otimes_U \mathrm{Tor}^V_q(U,U) \to R$$

If $J = \ker \varphi$, $J/J^2 = \mathrm{Tor}_1^V(U,U) \cong \Omega^1_{S/k} \otimes_S U$

$$\Rightarrow \Omega^q_{S/R} \otimes_S U \xrightarrow{\sim} \Lambda^q(\Omega^1 \otimes_S U) \xrightarrow{\sim} \Lambda^q(J/J^2) \to \mathrm{Tor}^q(U,U)$$

$$\Rightarrow \mathrm{Ext}^q_S(U,S) \otimes_S \Omega^q \to \mathrm{Ext}^q_S(U,U) \otimes_S \Omega^q \to R$$

assume S has ideal I s.t. S/I^n

f.g. proj. over R $\forall n$

then $H^q_I(S) = \varinjlim \mathrm{Ext}^q_S(S/I^n, S)$

above maps (for $U = S/I^n$) fit together to give

$$\left[\ \mathrm{res}^q_I : H^q_I(\Omega^q_{S/R}) = H^q_I(S) \otimes \Omega^q_{S/R} = \varinjlim \mathrm{Ext} \otimes \Omega \to R\ \right]$$

<u>Ex.</u> $S = R[[T_1, \ldots, T_q]]$, $I = (T_1, \ldots, T_q)$

Calculate $H^q_I(\Omega)$ by Cech complex.

$$\mathrm{res}_q \left(\sum_N \frac{a_N T^N}{T_1^{n_1} \cdots T_q^{n_q}} \right) = a_{(n_1 - 1, \ldots, n_q - 1)}$$

(note: the above denominator is fixed, indep. of N)

Globalize: Describe res_q in terms involving formal scheme data which globalize:

$$\begin{tikzcd} \mathrm{Spf}(S) \arrow[r, "\kappa_x"] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spf}(R) \arrow[r, "\kappa_y"] & \mathcal{Y} \end{tikzcd}$$

(proper)

x closed pt. of $\mathcal{X}$

$y = f(x)$ .... of $\mathcal{Y}$

$S = \hat{\mathcal{O}}_{\mathcal{X},x}$, $R = \hat{\mathcal{O}}_{\mathcal{Y},y}$

Want data to make sense for all terms and "commute" with κ's

Ingredients (5 operations on derived categories)

f_* : restriction-of-scalars (derived direct image)

f^* : left-adjoint

$f^{\#}$: right adjoint for "torsion" $\circ f_*$

$E \underline{\underline{\otimes}} F$: $\mathbb{R}\,Hom(E,F) =: [E,F]$

$Hom(E \underline{\underline{\otimes}} F, G) \cong Hom(E, [F,G])$ "closed category"

FLAT BASE-CHANGE

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\mu"'] & S \arrow[d, "\nu"] \\ \mathcal{U} \arrow[r, "\xi"'] & \mathcal{V} = S\hat{\otimes}_R \mathcal{U} \end{tikzcd}$$

(adic rings, ξ, ν canonical maps)

Assume μ (hence ν) flat. Then the map

$$\mu^*\varphi_* \to \mu^*\varphi_*\nu_*\nu^* = \mu^*\mu_*\xi_*\nu^* \to \xi_*\nu^*$$

is iso.

Can then define base-change $\beta: \nu^*\varphi^\# \to \xi^\#\mu^*$ dual to

$$\xi_*(R\Gamma'\nu^*\varphi^\#) \to R\Gamma'_{\mathcal{U}}\xi_*\nu^* R\Gamma'_S\varphi^\#$$

$$\cong R\Gamma'_{\mathcal{U}}\mu^*\varphi_* R\Gamma'_S\varphi^\# \to R\Gamma'_{\mathcal{U}}\mu^* \to \mu^*$$

Thm (Assume φ pseudofinite) For f.g. R-modules M, $\beta(M)$ is iso.

Idea Translate into relation between derived Hom and $\otimes$ ...
Use Greenlees-May duality: $1^\# M = M$

Global analogue (formal schemes) harder.

FUNDAMENTAL CLASS (FLAT MAPS)

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & S \arrow[d, shift left, "\nu"] \\ S \arrow[r, "\xi"'] & V = S\hat{\otimes}_R S \arrow[u, shift left, "\rho"] \arrow[l, bend right, "\rho"'] \end{tikzcd}$$

$\rho = $ mpn, $\rho\nu = \rho\xi = 1$

<u>Define</u>

$f : \rho^*\rho_* S \longrightarrow \varphi^\# R$

(equivalently: $R\Gamma'_S \rho^*\rho_* S \to R$)

• and note that homology of $\rho^*\rho_* S$ is $\mathrm{Tor}^V_q(S,S)$

1. Have $E \to \xi^\#\xi_* E$ dual to

$$\xi_* R\Gamma'_V E \xrightarrow{\kappa} E \qquad (E \in D(V))$$

2. $$\begin{array}{c} \rho^*\rho_* S \xrightarrow{\alpha} \rho^*\xi^\#\xi_*\rho_* S = \rho^*\xi^\# S \\ \| \\ \rho^*\xi^\#\mu^* R \xrightarrow[\text{base change}]{\sim} \rho^*\nu^*\varphi^\# R \\ \| \\ \varphi^\# R \end{array}$$

Now prove ...

FORMAL UNDERPINNING

Some axioms • $\exists$ functorial $\varphi_* E \otimes \varphi_* F \to \varphi_*(E \otimes F)$

• $\exists\ (\varphi\psi)_* \xrightarrow{\sim} \varphi_* \psi_*$ (monoidal pseudofunctor)

• $\varphi_* E \otimes F \to \varphi_* E \otimes \varphi_* \varphi^* F \to \varphi_*(E \otimes \varphi^* F)$ is iso (projection iso).

Write $[E,F] := R\mathcal{H}om^{\cdot}(E,F)$, so have iso

$$(*) \qquad \mathrm{Hom}(E' \otimes E, F) \xrightarrow{\sim} \mathrm{Hom}(E, [E', F]) \qquad \text{closed category}$$

Define iso $\qquad \varphi^\#[F,E] \xrightarrow{\sim} [\varphi^* F, \varphi^\# E]$

conjugate to $\quad \varphi_*((G \otimes \varphi^* F) \otimes R\Gamma' s) \xrightarrow{\sim} \varphi_*((G \otimes R\Gamma' s) \otimes \varphi^* F)$

$$\xrightarrow{\sim} \varphi_*(G \otimes R\Gamma' s) \otimes F$$

LEMMA The map $f' : S \to [\rho^* \rho_* S, \varphi^\# R]$ corresponding under $(*)$ to $f : S \otimes_{\mathcal{O}_S} \rho^* \rho_* S \to \varphi^\# R$ factors naturally as

$$S \to [\varphi^\# R, \varphi^\# R] \to [\rho^* \nu^* \varphi^\# R, \varphi^\# R] \to [\rho^* \rho_* S, \varphi^\# R]$$

e.g., Lemma 1.7.12):

(1.8.7a)
$$\begin{tikzcd} W \arrow[r, "\xi_1"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\sigma_1"] & W_1 \arrow[r, "\xi_2"] \arrow[d, "\nu_1"] \arrow[dr, phantom, "\sigma_2"] & X \arrow[d, "\mu"] \\ Y \arrow[r, "\varphi_1"'] & Y_1 \arrow[r, "\varphi_2"'] & Z \end{tikzcd}$$

(1.8.7b)
$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\nu_1"'] \arrow[dr, phantom, "\sigma_1"] & X \arrow[d, "\mu_1"] \\ W_1 \arrow[r, "\xi_1"'] \arrow[d, "\nu_2"'] \arrow[dr, phantom, "\sigma_2"] & X_1 \arrow[d, "\mu_2"] \\ Y \arrow[r, "\varphi"'] & Z \end{tikzcd}$$

PROPOSITION 1.8.8. *In the preceding situation, let $\boldsymbol{b}_0$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ be the base-change maps associated to σ_0, σ_1 and σ_2 respectively. Then with reference to diagram* (1.8.7a) (resp. (1.8.7b)), $\boldsymbol{b}_0$ *factors as*

(1.8.8a) $$\nu^*(\varphi_2\varphi_1)^\# \underset{(1.6.1)}{\xrightarrow{\sim}} \nu^*\varphi_1^\#\varphi_2^\# \underset{\boldsymbol{b}_1}{\longrightarrow} \xi_1^\#\nu_1^*\varphi_2^\# \underset{\xi_1^\#\boldsymbol{b}_2}{\longrightarrow} \xi_1^\#\xi_2^\#\mu^* \underset{(1.6.1)}{\xrightarrow{\sim}} (\xi_2\xi_1)^\#\mu^*, \quad \text{resp.}$$

(1.8.8b) $$(\nu_2\nu_1)^*\varphi^\# \underset{\text{nat'l}}{\xrightarrow{\sim}} \nu_1^*\nu_2^*\varphi^\# \underset{\nu_1^*\boldsymbol{b}_2}{\longrightarrow} \nu_1^*\xi_1^\#\mu_2^* \underset{\boldsymbol{b}_1}{\longrightarrow} \xi^\#\mu_1^*\mu_2^* \underset{\text{nat'l}}{\xrightarrow{\sim}} \xi^\#(\mu_2\mu_1)^*.$$

PROOF. (At heart, the same as that of [**DFS**, p. 79, Lemma 7.5.2].)

(a) In diagram (1.8.8.1), in which $(A, A_1, B, C, C_1) := (A_Y, A_{Y_1}, A_X, A_W, A_{W_1})$, the maps incorporate (combinations of) basic ones such as (1.4.12), (1.4.13), (1.6.1), (1.4.21) and (1.4.15), occurrences of which may not be explicitly indicated, but should nevertheless be apparent. Going from the top left corner of this diagram to the bottom right corner in the counterclockwise direction gives, by definition, the map dual (via (1.4.14)) to $\boldsymbol{b}_0$, while the clockwise direction gives the dual of (1.8.8a). (To verify the latter use that the bottom row of the commutative diagram (1.8.8.2) is the factorization of the trace map $\operatorname{tr}_{\xi_2\xi_1}\colon (\xi_2\xi_1)_*((\xi_2\xi_1)^\#\mu^*\otimes C) \to \mu^*$ given by Corollary 1.6.3.) Hence to show that (1.8.8a) is $\boldsymbol{b}_0$ it's enough to show that (1.8.8.1) commutes. In that diagram:

Commutativity of subrectangle $\square_1$ follows from uniqueness in Lemma 1.4.19(i).

Commutativity of $\square_2$ is left as an exercise, involving the "transitivity" relation between the projection maps for ξ_1, ξ_2, and $\xi_1\xi_2$ given by (1.4.23).

Commutativity of $\square_3$, left as an exercise, involves transitivity of θ (1.7.17).

Since $\xi_{1*}\mathbf{D}_C \subset \mathbf{D}_{C_1}$ (see Lemma 1.4.19(iii)$'$), commutativity of $\square_4$ can be verified *after* composing with the natural map $\xi_{2*}(\nu_1^*\varphi_2^\#\otimes C_1) \to \xi_{2*}\nu_1^*\varphi_2^\#$ (cf. (1.1.8.1)), at which point one need only refer to the definition of $\boldsymbol{b}_1$.

Commutativity of $\square_5$ is given directly by the definition of $\boldsymbol{b}_2$.

Commutativity of $\square_6$ is given by transitivity of traces (Corollary 1.6.3).

Commutativity of the remaining subrectangles is clear.

(1.8.8.1)

$$
\begin{tikzcd}
(\xi_2\xi_1)_*(\nu^*(\varphi_2\varphi_1)^\# \otimes C) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\square_1"] & \xi_{2*}\xi_{1*}(\nu^*\varphi_1^\#\varphi_2^\# \otimes C) \arrow[r, "b_1"] \arrow[d] & \xi_{2*}\xi_{1*}(\xi_1^\#\nu_1^*\varphi_2^\# \otimes C) \arrow[r, "\sim", "(1.4.19)(\mathrm{iii})'"'] & \xi_{2*}(\xi_{1*}(\xi_1^\#\nu_1^*\varphi_2^\# \otimes C) \otimes C_1) \arrow[dd, "\mathrm{tr}_{\xi_1}"] \\
(\xi_2\xi_1)_*(\nu^*(\varphi_2\varphi_1)^\# \otimes \nu^*A \otimes (\xi_2\xi_1)^*B) \arrow[d, "\simeq"] & \xi_{2*}\xi_{1*}(\nu^*\varphi_1^\#\varphi_2^\# \otimes \nu^*A \otimes \xi_1^*C_1) \arrow[l, "(1.4.19)(\mathrm{ii})"'] \arrow[d, "\simeq"', "\theta_{\sigma_1}^{-1}"] \arrow[rr, phantom, "\square_4"] & & \\
\xi_{2*}\xi_{1*}(\nu^*(\varphi_2\varphi_1)^\# \otimes \nu^*A) \otimes B \arrow[d, "\theta_{\sigma_1}^{-1}"'] \arrow[r, phantom, "\square_2"] & \xi_{2*}(\nu_1^*\varphi_{1*}(\varphi_1^\#\varphi_2^\# \otimes A) \otimes C_1) \arrow[rr, "\mathrm{tr}_{\varphi_1}"] \arrow[d] & & \xi_{2*}(\nu_1^*\varphi_2^\# \otimes C_1) \arrow[d, equal] \\
\xi_{2*}(\nu_1^*\varphi_{1*}(\varphi_1^\#\varphi_2^\# \otimes A) \otimes \xi_2^*B) \arrow[d, "\theta_{\sigma_1}"'] & \xi_{2*}(\nu_1^*\varphi_{1*}(\varphi_1^\#\varphi_2^\# \otimes A) \otimes \nu_1^*A_1 \otimes \xi_2^*B) \arrow[l, "\sim"', "(1.4.19)(\mathrm{iii})'"] \arrow[r, "\mathrm{tr}_{\varphi_1}"] \arrow[d, "\simeq"', "\theta_{\sigma_2}^{-1}"] & \xi_{2*}(\nu_1^*\varphi_2^\# \otimes \nu_1^*A_1 \otimes \xi_2^*B) \arrow[d, "\theta_{\sigma_2}^{-1}"', "\simeq"] & \xi_{2*}(\nu_1^*\varphi_2^\# \otimes C_1) \arrow[l] \arrow[d, "b_2"] \\
\xi_{2*}\xi_{1*}(\nu^*(\varphi_2\varphi_1)^\# \otimes \nu^*A) \otimes B \arrow[d, "\simeq"] \arrow[r, phantom, "\square_3"] & \mu^*\varphi_{2*}(\varphi_{1*}(\varphi_1^\#\varphi_2^\# \otimes A) \otimes A_1) \otimes B \arrow[r, "\mathrm{tr}_{\varphi_1}"] \arrow[d, "\simeq"', "(1.4.19)(\mathrm{iii})'"] \arrow[dr, phantom, "\square_6"] & \mu^*\varphi_{2*}(\varphi_2^\# \otimes A_1) \otimes B \arrow[d, "\mathrm{tr}_{\varphi_2}"'] \arrow[r, phantom, "\square_5"] & \xi_{2*}(\xi_2^\#\mu^* \otimes C_1) \arrow[d, "\mathrm{tr}_{\xi_2}"] \\
(\xi_2\xi_1)_*\nu^*((\varphi_2\varphi_1)^\# \otimes A) \otimes B \arrow[r, "\sim", "\theta_{\sigma_0}^{-1}"'] & \mu^*(\varphi_2\varphi_1)_*((\varphi_2\varphi_1)^\# \otimes A) \otimes B \arrow[r, "\mathrm{tr}_{\varphi_1\varphi_2}"'] & \mu^* \otimes B \arrow[r] & \mu^*
\end{tikzcd}
$$

(1.8.8.2)

$$
\begin{tikzcd}
(\xi_2\xi_1)_*(\xi_1^\#\nu_1^*\varphi_2^\# \otimes C) \arrow[r, equal] \arrow[d, "b_2"'] & \xi_{2*}\xi_{1*}(\xi_1^\#\nu_1^*\varphi_2^\# \otimes C) \arrow[r, "\sim", "(1.4.19)(\mathrm{iii})'"'] \arrow[d, "b_2"'] & \xi_{2*}(\xi_{1*}(\xi_1^\#\nu_1^*\varphi_2^\# \otimes C) \otimes C_1) \arrow[r, "\mathrm{tr}_{\xi_1}"] \arrow[d, "b_2"] & \xi_{2*}(\nu_1^*\varphi_2^\# \otimes C_1) \arrow[d, "b_2"] \\
(\xi_2\xi_1)_*(\xi_1^\#\xi_2^\#\mu^* \otimes C) \arrow[r, equal] \arrow[d, "(1.6.1)"', "\simeq"] & \xi_{2*}\xi_{1*}(\xi_1^\#\xi_2^\#\mu^* \otimes C) \arrow[r, "\sim", "(1.4.19)(\mathrm{iii})'"'] \arrow[d, "(1.6.1)"', "\simeq"] & \xi_{2*}(\xi_{1*}(\xi_1^\#\xi_2^\#\mu^* \otimes C) \otimes C_1) \arrow[r, "\mathrm{tr}_{\xi_1}"'] & \xi_{2*}(\xi_2^\#\mu^* \otimes C_1) \arrow[d, "\mathrm{tr}_{\xi_2}"] \\
(\xi_2\xi_1)_*((\xi_2\xi_1)^\#\mu^* \otimes C) \arrow[r, equal] & \xi_{2*}\xi_{1*}((\xi_2\xi_1)^\#\mu^* \otimes C) \arrow[r, "\sim", "(1.4.19)(\mathrm{iii})'"'] & \xi_{2*}(\xi_{1*}((\xi_2\xi_1)^\#\mu^* \otimes C) \otimes C_1) \arrow[u, "\simeq", "(1.6.1)"'] \arrow[r, dashed] & \mu^*
\end{tikzcd}
$$