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"Some applications of algebraic K-theory in commutative algebra"

Thomason localization

Affine normal surfaces / $k = \bar{k}$

$X = \operatorname{Spec}(A)$, A: affine domain of dim $d / k = \bar{k}$

Say A has the complete intersection property if every smooth maximal ideal $\mathfrak{m}$ of A (of height $d = \dim A$) is a complete intersection in A.

joint with Amalendu Krishna.

<u>Thm. 1.</u> Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective curve, and let A = homogeneous coord. ring of X. Then A has the complete intersection property $\Leftrightarrow H^1(X, \mathcal{O}_X(1)) = 0$

$$\left(\Leftrightarrow H^0(X, \omega_X(-1)) = 0 \right)$$

<u>Ex.</u> If $X \subset \mathbb{P}^n_{\mathbb{C}}$, non-degenerate (not in a hyperplane) and $\deg X \leq 2n-1$, then $A(X)$ has the c.i. property (by Clifford's thm, $H^1(X, \mathcal{O}_X(1)) = 0$).

<u>Thm. 2.</u> Let $X = \operatorname{Spec} A$ be a normal affine surface $/\mathbb{C}$, connected. The A has the c.i. property $\iff$

(i) X is a Zariski open subset of a projective surface $\overline{X}/\mathbb{C}$ with $H^2(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$.

(ii) $\exists f \neq 0$ on A s.t. A_f has the c.i. property.

<u>Rmk.</u> Suppose $\overline{X}$ as in Thm. 2 and $\pi: Y \to \overline{X}$ is a resolution of sing's. Then $H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \twoheadrightarrow H^2(Y, \mathcal{O}_Y)$, so if $H^2(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$, then $p_g(Y) = 0$.

The Bloch conjecture asserts that for

smooth proj. Y with $p_g(Y) = 0$, any affine open has the c.i. property.

Ex Let $X = \operatorname{Spec} A$, where A is normal with quotient field $\mathbb{C}(x,y)$. Then A has the c.i. property $\Leftrightarrow$ X has rational singularities.

Thm. 3. Let A be a 2-dim, normal, $\mathbb{N}$-graded $\overline{\mathbb{Q}}$-algebra of dim 2. Then A has the c.i. property; all proj. A-modules are free.

Ex $A = \overline{\mathbb{Q}}[x,y,z]/x^n+y^n+z^n$, $n \geq 4$

Then A has the c.i. property

$A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ does not have the c.i. property; a smooth maximal ideal of $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is a c.i. $\Leftrightarrow$ it contains a homog. element of A of positive degree.

Rmk A conjecture of Bloch + Beilinson implies that any smooth affine domain of dim > 1 over $\overline{\mathbb{Q}}$ has the complete intersection property. This is interesting mainly if $A \otimes \mathbb{C}$ fails to have the c.i. property.

Suggest a similar conj. for arbitrary affine domains of dim ≥ 2 over $\overline{\mathbb{Q}}$.

Ex $A_k = k[x, y, z]/xyz(1-x-y-z)$

The c.i. property for A_k $(k = \bar{k})$ is equiv. to $K_2(k) = 0$.

$\therefore$ $A_{\overline{\mathbb{Q}}}$ has the c.i. property

($K_2(\overline{\mathbb{Q}}) = 0$, Garland; Borel)

$$\longrightarrow \overline{\mathbb{Q}}^* \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}^* / \langle a \otimes 1 - a \mid a \in \overline{\mathbb{Q}}^* \setminus \{1\} \rangle$$

$A_{\mathbb{C}}$ does not have it ($K_2(\mathbb{C})$ very large).

Thm. 4. (Murthy) Any affine domain / $\overline{\mathbb{F}_p}$ of dim > 1 has the c.i. property.

Thm. Let X/k be a normal proj. surface, $\pi : Y \to X$ a resolution of sing's.
E = exceptional locus. Then there are isom.'s

$$CH^2(X) \ (= F^2 K_0(X)) \xrightarrow{\sim} \varprojlim_n F^2 K_0(Y, nE)$$

$$\cong F^2 K_0(Y, nE) \qquad n \gg 0.$$

$K_i(X) = \pi_{i+1}(BQ\,\mathbb{P}(X))$

$nE \hookrightarrow Y$ $\qquad$ $F(BQ\,\mathbb{P}(Y) \to BQ\,\mathbb{P}(nE))$

= homotopy fiber.

joint with P. Roberts

example of Dutta, Hochster, MacLaughlin

$$R = k[x,y,z,w]_{(x,y,z,w)} \Big/ xy - zw$$

$N = R/(x,z)$, $M =$ a certain module of finite length + finite proj. dim.

$$\chi_R(M,N) = \sum_{i \geq 0} (-1)^i \, \ell_R\left(\mathrm{Tor}_i^R(M,N)\right) < 0 .$$

R reg. local; M, N finite R-modules s.t.
$\ell(M \otimes N) < \infty$, $\dim M + \dim N < \dim R$,
then $\chi(M,N) = 0$.
(Serre; if R is equichar P. Roberts + (Gillet & Soule))

R Noeth. local ring
$\mathcal{C}$ = cat. of bounded complexes of free R-mod's of finite rank with cohomology of finite length

$K_0(\mathcal{C})$ = Grothendieck group
$G_0(R)$ = Grothendieck group of fin. gen. R-modules

$$\chi : K_0(\mathcal{C}) \otimes_{\mathbb{Z}} G_0(R) \longrightarrow \mathbb{Z}$$

$$K_0(\mathcal{C})_{\mathbb{Q}} \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(G_0(R)_{\mathbb{Q}}, \mathbb{Q})$$

$$G_0(R)_{\mathbb{Q}} \underset{\tau}{\cong} CH_*(R)_{\mathbb{Q}}$$

The local Chern character + local Riemann-Roch give rise to a comm. diag.

$$\begin{tikzcd} K_0(\mathcal{C})_{\mathbb{Q}} \otimes G_0(R)_{\mathbb{Q}} \arrow[rr, "1\otimes\tau"] \arrow[dr, "\chi"'] & & K_0(\mathcal{C})_{\mathbb{Q}} \otimes CH_*(R)_{\mathbb{Q}} \arrow[dl, "ch"] \\ & \mathbb{Q} & \end{tikzcd}$$

R = vertex ring associated to the cone over a smooth proj. variety $X \subset \mathbb{P}^n_k$, $k = \bar{k}$

$$CH_*(R)_{\mathbb{Q}} = CH^*(R)_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}} / \langle h \rangle$$

$CH^*(X)_{\mathbb{Q}} \longrightarrow CH^*_{num}(X)_{\mathbb{Q}}$ = cycles modulo numerical equiv.

Let $W = \ker\left(CH^*(X)_{\mathbb{Q}} \xrightarrow{\cdot h} CH^*(X)_{\mathbb{Q}} \right)$

$$V = \mathrm{image}(W) \subset CH^*_{num}(X)_{\mathbb{Q}} \cong \mathrm{Hom}(CH^*_{num}(X)_{\mathbb{Q}}, \mathbb{Q}) \hookrightarrow \mathrm{Hom}(CH^*(X)_{\mathbb{Q}}, \mathbb{Q})$$

$$\cup$$

Thm. $V = \mathrm{image}\, K_0(\mathcal{C})_{\mathbb{Q}}$ $\qquad \mathrm{Hom}(CH^*(X)_{\mathbb{Q}}/\langle h\rangle, \mathbb{Q})$

$$\|$$

$$\mathrm{image}\, K_0(\mathcal{C})_{\mathbb{Q}} \hookrightarrow \mathrm{Hom}(CH_*(R)_{\mathbb{Q}}, \mathbb{Q})$$

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

$$\ulcorner CH^*(X)_{\mathbb{Q}} \cong H^*(X, \mathbb{Q}) = CH^*_{num}(X)_{\mathbb{Q}} \lrcorner$$

$$\dim\left(H^*(X, \mathbb{Q}) \xrightarrow{c(h)} H^*(X, \mathbb{Q})\right) > 1$$

Standard Conj. (Grothendieck)

$$CH^*_{num}(X)_{\mathbb{Q}} \hookrightarrow H^*(X_{\mathbb{C}}, \mathbb{Q}) \quad \text{if } k = \bar{k} \subset \mathbb{C}$$

$$\ker\left(h : CH^*_{num}(X)_{\mathbb{Q}} \to CH^*_{num}(X)_{\mathbb{Q}}\right)$$