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"Numerical equivalence on Chow groups of local rings"

R: Noetherian local ring.

$C(R)$: the category of bounded complexes $\mathbb{F}_.$
$\underset{\mathcal{C}}{\shortparallel}$ s.t. each F_i is finite R-free, $\ell_R(H_i(\mathbb{F}_.)) < \infty \quad \forall i$

$$G_0(R)_{\mathbb{Q}} := \bigoplus_{M:\text{f.g. } R\text{-mod.}} \mathbb{Q}[M] \Big/ \left\langle [M]-[L]-[N] : 0 \to L \to M \to N \to 0 \text{ exact} \right\rangle_{\mathbb{Q}}$$

$\mathbb{F}_. \in C(R)$; $\chi_{\mathbb{F}_.} : G_0(R)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$

$$[M] \longmapsto \sum_i (-1)^i \ell_R(H_i(\mathbb{F}_. \otimes M))$$

Def. $NG_0(R)_{\mathbb{Q}} := \{ c \in G_0(R)_{\mathbb{Q}} : \chi_{\mathbb{F}_.}(c) = 0 \quad \forall \mathbb{F}_. \in C(R) \}$

$NG_0(R)_{\mathbb{Q}} \subseteq G_0(R)_{\mathbb{Q}}$, c : numerically equiv. to 0.

$$\overline{G_0(R)_{\mathbb{Q}}} := G_0(R)_{\mathbb{Q}} \Big/ NG_0(R)_{\mathbb{Q}}$$

<u>Thm.</u> R: excellent local ring s.t.

$$\begin{cases} R \supset \mathbb{Q} & \text{or} \\ R \text{ is essentially of finite type / a field, } \mathbb{Z}\text{, or complete DVR} \end{cases}$$

$$\Longrightarrow \dim_{\mathbb{Q}} \overline{G_0(R)_{\mathbb{Q}}} < \infty$$

$$\| $$

$$\dim \operatorname{Im}\left(K_0(\mathcal{C}(R)) \longrightarrow \operatorname{Hom}(G_0(R)_{\mathbb{Q}}, \mathbb{Q})\right)$$

<u>examples</u>

① $\mathbb{K}_\cdot$: Koszul complex of a s.o.p. $\underline{a}$ of R

$$\chi_{\mathbb{K}_\cdot} : \begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \longrightarrow & \mathbb{Q} \\ \Psi & & \Psi \\ [M] & \longmapsto & e_{(\underline{a})}(M) \end{array}$$

$NG_0(R)_{\mathbb{Q}} \subsetneq G_0(R)_{\mathbb{Q}}$

- $\dim M < \dim R \Rightarrow \chi_{\mathbb{K}_\cdot}(M) = 0$
- $\dim M = \dim R \Rightarrow \chi_{\mathbb{K}_\cdot}(M) > 0$

$\dim \overline{G_0(R)_{\mathbb{Q}}} > 0$

$\left(R: \text{RLR} \Rightarrow \dim \overline{G_0(R)_{\mathbb{Q}}} = 1\right)$

② (Dutta – Hochster – MacLaughlin, Levin, Miller – Singh, Roberts – Srinivas)

k : field, $R_{m,n} := k[x_{ij} \mid i = 1..m, j = 1..n]_{(x)} \Big/ I_2(x_{ij})$.

$(\dim R_{m,n} = m+n-1, \quad \mathrm{Proj}\, R_{m,n} = \mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$.

$2 \le m \le n \Rightarrow$ for $i = n, n+1, \ldots, n+m-1$

$\exists\ \mathbb{H}_\cdot \in C(R_{m,n})$ s.t.

$$\begin{cases} \chi_{\mathbb{H}_\cdot}(M) = 0 & \text{if } \dim M < i \\ \chi_{\mathbb{H}_\cdot}(N) \neq 0 & \text{for } \dim N = i \end{cases}$$

$\underline{\underline{NG_0(R)_{\mathbb{Q}} = 0}}$ $\qquad G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} \overline{G_0(R)_{\mathbb{Q}}} = \mathbb{Q}^m$

③ $T := \mathbb{C}[x,y,z] \Big/ \langle y^2 z - x^3 + x z^2 \rangle$

$R := T_{(x,y,z)}$ $\qquad G_0(R)_{\mathbb{Q}} = \mathbb{Q} \oplus (E \otimes_{\mathbb{Z}} \mathbb{Q})$

$E := \mathrm{Proj}(T)$ $\qquad \dim G_0(R)_{\mathbb{Q}} = \infty$

↳ abelian group

$\mathfrak{p}$: homog. prime ideal of T of $ht = 1$

(i) $\chi_{\mathbb{F}_\cdot}(R/\mathfrak{p}R) = 0$ for $\mathbb{F}_\cdot \in C(R)$

(by a theorem due to P. Roberts)

(ii) $[R/\mathfrak{p}R] \neq 0$ in $G_0(R)_{\mathbb{Q}}$ $\Longleftrightarrow$

$\mathfrak{p} \in E$ is of order ∞.

$\dim \overline{G_0(R)_{\mathbb{Q}}} = 1$.

<u>Remark</u> $R \longrightarrow S$ (inj.) finite local homo.

$\Longrightarrow$ $\overline{G_0(R)_{\mathbb{Q}}} \longleftarrow \overline{G_0(S)_{\mathbb{Q}}}$ (surj.)

($\Longrightarrow$) $\dim \overline{G_0(R)_{\mathbb{Q}}} \leq \dim \overline{G_0(S)_{\mathbb{Q}}}$

<u>Application</u>

<u>Def.</u> $(R; \mathfrak{m})$ local ring, $\operatorname{char}(R) = p > 0$, $\dim R = d$.

F-finite, $\overline{R/\mathfrak{m}} = R/\mathfrak{m}$

$$F : R \longrightarrow \underset{\displaystyle F_R}{\overset{\displaystyle R}{\shortparallel}}$$

R: numerically Roberts ring $\iff$

$$[{}^F R] - p^d [R] \in NG_0(R)_{\mathbb{Q}}$$

(Roberts ring $:\iff$ $[{}^F R] - p^d [R] = 0 \in G_0(R)_{\mathbb{Q}}$)

R: num. Roberts ring $\Rightarrow [{}^{F^n} R] - p^{nd}[R] \in NG_0(R)_{\mathbb{Q}}$

$\Longrightarrow \forall \mathbb{F}. \in C(R)$: $\chi_{\mathbb{F}.}({}^{F^n} R) = p^{nd} \chi_{\mathbb{F}.}(R)$

$$\lim_{n\to\infty} \frac{\chi_{\mathbb{F}.}({}^{F^n} R)}{p^{nd}} = \chi_{\mathbb{F}.}(R)$$

$\|$

$\chi_\infty(\mathbb{F}.)$ Dutta multiplicity.

<u>Thm.</u> R : num. Roberts ring $\iff \chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(R)$
$\forall \mathbb{F}. \in C(R)$

if R is CM $\iff e_{HK}(I) = \ell_R(R/I)$
$\forall I$ m-primary s.t. $pd(I) < \infty$.

complete intersection $\Downarrow$ Roberts ring

Gorenstein $\not\Downarrow$ num. Roberts ring (Miller - Singh 2000) $\hookrightarrow$ dim = 5

Roberts ring $\Longrightarrow$ num. Roberts ring

num. Roberts ring $\Downarrow$ vanishing theorem

Roberts ring $\Longrightarrow$ vanishing theorem (Roberts '85)

quot. sing. $\Longrightarrow$ Roberts ring

Galois ext. of a RLR $\Longrightarrow$ Roberts ring

homog. coord. ring of abelian variety $\Longrightarrow$ Roberts ring

- $\dim R \leq 2$ $\Longrightarrow$ num. Roberts ring
  R: equi-dim $\not\Longrightarrow$ Roberts ring
- $\dim R = 3$ $\Longrightarrow$ num. Roberts ring
  R: Gor.
- $R_{m,n}$ num. Roberts ring $\Longleftrightarrow$ $R_{m,n}$ Roberts ring $\Longleftrightarrow$ $m = n = 2$

<u>Fact</u> R : Roberts ring $\Longrightarrow$ R_p : Roberts ring

<u>Question</u> R : num. Roberts ring $\overset{?}{\Longrightarrow}$ is R_p as well?

<u>Rmk</u> By singular Riemann - Roch theorem (Baum - Fulton - MacPherson)

$$\exists \ \tau_R : G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} A_*(R)_{\mathbb{Q}}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr$ chow group

$$NA_*(R)_{\mathbb{Q}} := \tau_R\left(NG_0(R)_{\mathbb{Q}}\right)$$

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \overline{G_0(R)_{\mathbb{Q}}} & \xrightarrow{\overline{\tau_R}} & \overline{A_*(R)_{\mathbb{Q}}} := A_*(R)_{\mathbb{Q}} / NA_*(R)_{\mathbb{Q}} \\ & & \| \\ & & \bigoplus_i \overline{A_*(R)_{\mathbb{Q}}} \quad \text{(i.e. graded)} \end{array}$$

Enough to show: $\dim \overline{A_*(R)_{\mathbb{Q}}} < \infty$

<u>Pf. of thm.</u>

step 1. $\xrightarrow{\text{reduce}}$ R : domain

step 2. take a regular alteration of Spec R

<u>Thm.</u> R : excellent local ring s.t.

$$\begin{cases} R \supseteq \mathbb{Q} \quad \text{or} \\ R \text{ is ess. of finite type / a field, } \mathbb{Z}, \text{ or complete DVR} \end{cases}$$

$\Longrightarrow \dim \overline{A_*(R)_{\mathbb{Q}}} < \infty.$

step 2. take a regular alteration of $\mathrm{Spec}(R)$ $\left(\begin{array}{l}\text{Hironaka}\\ \text{de Jong}\end{array}\right)$

$$Z \xrightarrow{f} \mathrm{Spec}(R) \quad \text{s.t.}$$

$$\left\{\begin{array}{l} Z : \text{ regular scheme} \\ f : \text{ projective, surjective} \\ \text{gen. finite } \left(\text{i.e. } [R(Z) : Q(R)] < \infty\right) \end{array}\right.$$

May assume that $f^{-1}(m)_{red}$ is SNC

$$\begin{array}{c} f^{-1}(m)_{red} \\ \| \\ E_1 \cup \dots \cup E_t \end{array} \quad (\rightarrow E_i \text{ is reg. } \forall i)$$

$$\begin{tikzcd}
A_*(Z)_\mathbb{Q} \arrow[r, "f_*"] & A_*(R)_\mathbb{Q} \arrow[l, dotted, bend left, "\exists s"]
\end{tikzcd} \qquad f_* \circ s = id.$$

$$\begin{tikzcd}
A_*(R)_\mathbb{Q} \arrow[r, "s"] \arrow[d] & A_*(Z)_\mathbb{Q} \arrow[rr, "\sum_j E_j"] \arrow[dr, two heads] \arrow[drr, "\varphi"] & & \bigoplus_j A_*(E_j)_\mathbb{Q} \arrow[d] \\
\overline{A}_*(R)_\mathbb{Q} & & \mathrm{Im}(\varphi) \arrow[ll, dashed, two heads, "\exists"'] \arrow[r, phantom, "\subset"] & \bigoplus_j CH_{num}(E_j)_\mathbb{Q}
\end{tikzcd}$$

$$CH_{num}(\ )_\mathbb{Q} = \frac{\text{chow ring}}{\sim \text{num. equiv.}}, \quad \dim_\mathbb{Q} CH_{num}(\ )_\mathbb{Q} < \infty$$

Claim: $\ker \varphi \subset NA_*(R)_{\mathbb{Q}}$

<u>Rmk</u> $A := \bigoplus_{n \geq 0} A_n = \mathbb{C}[A_1]$, $\quad M = A_+$

Assume $X := \mathrm{Proj}(A)$ is smooth$/\mathbb{C}$

$R := A_M$

$M \neq P$: homog. prime of A

$$
\begin{tikzcd}
 & & & {[\mathrm{Proj}\, A/p]} \arrow[rr, mapsto] \arrow[d, phantom, "\cap"] & & {\{\mathrm{Spec}\, R/pR\}} \arrow[d, phantom, "\cap"] & \\
0 \arrow[r] & W \arrow[r] \arrow[dl] \arrow[dd] & CH(X)_{\mathbb{Q}} \arrow[r, "h"] \arrow[dd] & CH(X)_{\mathbb{Q}} \arrow[rr] \arrow[dd] & & A_*(R)_{\mathbb{Q}} \arrow[r] \arrow[dd] \arrow[ddd, bend left] & 0 \\
V \arrow[dr, phantom, "\cap"] & & & & & & \\
0 \arrow[r] & L \arrow[r] & CH_{num}(X)_{\mathbb{Q}} \arrow[r, "h"] & CH_{num}(X)_{\mathbb{Q}} \arrow[rr] & & C \arrow[r] \arrow[d, dashed, "\exists \psi"'] & 0 \\
 & & & & & A_*(R)_{\mathbb{Q}} &
\end{tikzcd}
$$

<u>RS</u>

$$
\begin{array}{ccc}
\dim_{\mathbb{Q}} V = & \dim \overline{A_*(R)_{\mathbb{Q}}} & \left(= \dim \overline{G_0(R)_{\mathbb{Q}}}\right) \\
\cap & \cap & \\
\dim L = & \dim C &
\end{array}
$$

$V = L \iff C \xrightarrow{\Psi} \overline{A_*(R)_{\mathbb{Q}}}$ isom.

Esnault–Viehweg (RS) $\exists$ example s.t. $V \neq L$.

If $CH(X)_{\mathbb{Q}} \xrightarrow{\sim} CH_{num}(X)_{\mathbb{Q}}$

standard conj.

$\Downarrow$

$A_j(R)_{\mathbb{Q}} = 0$ if $j \leq \frac{\dim R}{2}$

$\Updownarrow$

$\forall\, \mathbb{F}_\cdot \in C(R)$
$\forall$ R-mod. N s.t. $\dim N \leq \frac{\dim R}{2}$
$\chi_{\mathbb{F}_\cdot}(N) = 0$

$\Downarrow$

$W = V = L$

$\Downarrow$

$\boxed{RS}$

$A_*(R)_{\mathbb{Q}} = C \xrightarrow{\sim} \overline{A_*(R)_{\mathbb{Q}}}$

i.e. $G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} \overline{G_0(R)_{\mathbb{Q}}}$

$NG_0(R_{m,n})_{\mathbb{Q}} = 0$

Question:
is it true for all R?

Gor. $d=4$
$\Downarrow$
num. Roberts ring

$\dim R \leq 3$
(Roberts)