COHOMOLOGICAL DEGREES

INTRODUCTION

Let $(R, \mathfrak{m})$ be a Noetherian local ring (or a Noetherian graded algebra) and let $\mathcal{M}(R)$ be the category of finitely generated R–modules (or the appopriate category of graded modules). A *degree function* is a numerical function $\mathbf{d} : \mathcal{M}(R) \mapsto \mathbb{N}$. The more interesting of them initialize on modules on finite length and have mechanisms that control how the functions behave under generic hyperplane section. Thus, for example, if L is a given module of finite length, one function may request

$$\mathbf{d}(L) := \lambda(L) = \text{length of } (L),$$

or in case that L be a graded module $L = \bigoplus_{i \in \mathbb{Z}} L_i$,

$$\mathbf{d}(L) := \sup\{i \mid L_i \neq 0\}.$$

How to deal with exact sequences of the form (generic hyperplane sections)

$$0 \to 0 :_A h \longrightarrow A \xrightarrow{\cdot h} A \longrightarrow A/hA \to 0, \quad \lambda(0 :_A h) < \infty,$$

become a fundamental issue.

Important Degrees

- **Castelnuovo-Mumford regularity** $\operatorname{reg}(\cdot)$: If h is a linear form of R whose annihilator $0 :_A h$ has finite length, then

$$\operatorname{reg}(A) = \max\{\operatorname{reg}(0 :_A h), \operatorname{reg}(A/hA)\}.$$

- **Classical multiplicity** $\deg(\cdot)$: Let A be a finitely generated R–module of dimension d and let $h \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that $0 :_A h$ has finite length. Then

$$\deg(A) = \begin{cases} \lambda(A) & \text{if } \dim A = 0 \\ \lambda(A/hA) - \lambda(0 :_A h) & \text{if } \dim A = 1 \\ \deg(A/hA) & \text{if } \dim A \geq 2 \end{cases}$$

These functions read different aspects of the module (in the words of Gunston, $\deg(\cdot)$ is “ranky”, $\operatorname{reg}(\cdot)$ is “shifty”) so it seems desirable to have functions with properties of both.

Arithmetic degree

If one wants to capture the contributions of each primary component of the module, adding them all gives rise to the **arithmetic degree** of M, $\text{arith-deg}(M)$. Assuming R is Gorenstein (for simplicity) it assembles itself into

$$\text{arith-deg}(M) = \sum_{i=0}^{n} \deg(\text{Ext}_R^i(\text{Ext}_R^i(M,R),R)).$$

$\text{arith-deg}(\cdot)$ has some of the properties of $\deg(\cdot)$, but the **Bertini's rule** goes out of whack:

$$\text{arith-deg}(M) \leq \text{arith-deg}(M/hM)$$

(one wants at least $\geq$).

BIG DEGS

In Gunston, Doering and myself introduced a general class of such functions. In his thesis, Tor Gunston carried out a more formal examination of such functions in order to introduce his own construction of a new cohomological degree. One of the points that must be take care of is that of an appropriate *generic hyperplane* section. Let us recall the setting.

Throughout we suppose that the residue field k of R is infinite.

Definition: If $(R,\mathfrak{m})$ is a local ring, a *notion of genericity* on $\mathcal{M}(R)$ is a function

$$U : \{\text{isomorphism classes of } \mathcal{M}(R)\} \longrightarrow \{\text{non-empty subsets of } \mathfrak{m} \setminus \mathfrak{m}^2\}$$

subject to the following conditions for each $A \in \mathcal{M}(R)$:

(i) If $f - g \in \mathfrak{m}^2$ then $f \in U(A)$ if and only if $g \in U(A)$.

(ii) The set $\overline{U(A)} \subset \mathfrak{m}/\mathfrak{m}^2$ contains a non-empty Zariski-open subset.

(iii) If depth$A > 0$ and $f \in U(A)$, then f in regular on A.

There is a similar definition for graded modules. We will usually switch notation, denoting the algebra by S.

Fixing a notion of genericity $U(\cdot)$ one has the following extension of the classical multiplicity:

Definition: A *cohomological degree*, or *extended multiplicity*, is a function

$$\operatorname{Deg}(\cdot) : \mathcal{M}(R) \mapsto \mathbb{N},$$

that satisfies

(i) If $L = \Gamma_{\mathfrak{m}}(A)$ is the submodule of elements of A which are annihilated by a power of the maximal ideal and $\overline{A} = A/L$, then

$$\mathrm{Deg}(A) = \mathrm{Deg}(\overline{A}) + \lambda(L),$$

where $\lambda(\cdot)$ is the ordinary length function.

(ii) (Bertini's rule) If A has positive depth and $h \in U(A)$, then

$$\mathrm{Deg}(A) \geq \mathrm{Deg}(A/hA).$$

(iii) (The calibration rule) If A is a Cohen–Macaulay module, then

$$\mathrm{Deg}(A) = \deg(A),$$

where $\deg(A)$ is the ordinary multiplicity of the module A.

For modules of $\dim M = 1$, there is only one value

$$\mathrm{Deg}(M) = \deg(M) + \lambda(\text{submodule of finite support}).$$

For $\dim R \geq 2$, there is an infinite number of Deg's.

Homological degree

Let M be a finitely generated graded module over the graded algebra R and let S be a Gorenstein graded algebra mapping onto R, with maximal graded ideal $\mathfrak{m}$. Assume that $\dim S = r$, $\dim M = d$. The **homological degree** of M is the integer

$$\begin{aligned} \operatorname{hdeg}(M) &= \deg(M) + \\ & \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \operatorname{hdeg}(\operatorname{Ext}_S^i(M,S)). \end{aligned}$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\begin{aligned} \operatorname{hdeg}(M) &= \deg(M) + \\ & \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \operatorname{hdeg}(\operatorname{Ext}_S^i(M,S)). \end{aligned}$$

If M is a module Cohen-Macaulay on the punctured spectrum, then

$$\operatorname{hdeg}(M) = \deg(M) + SV(M),$$

where $SV(M)$ is the Stuckrad-Vogel invariant of M. In particular, if A is a two dimensional standard graded algebra over a field, without embedded primes, then

$$\operatorname{Deg}(A) = \deg(A) + HR(A),$$

where $HR(A)$ is the Hartshorne-Rao number of A ($HR(A) = \lambda(\widetilde{A}/A)$, where $\widetilde{A}$ is the S_2-ification of A).

GENERAL PROPERTIES

Now we are going to see why these functions acquire their **big** degs designations ...

Let $\mathrm{Deg}(\cdot)$ be a cohomological degree. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay ring of dimension d, with an infinite residue field and let I be an ideal of codimension $g > 0$. If depth $R/I = r$, then

$$\begin{aligned} \nu(I) \quad &\leq \quad \deg(R) + (g-1)\mathrm{Deg}(R/I) \\ &+ \quad \underbrace{(d-g-r)(\mathrm{Deg}(R/I) - \deg(R/I))}_{\text{non-Cohen-Macaulay correction}} \end{aligned}$$

Another elementary observation: We assume that $(R,\mathfrak{m})$ is a Cohen–Macaulay local ring and denote its residue field by K. For any finitely generated R–module M, we denote by $\beta_i(M)$ its ith Betti number.

Let M be a finitely generated R–module. For any $\mathrm{Deg}(\cdot)$ function and any integer $i \geq 0$,

$$\beta_i(M) \leq \beta_i(K) \cdot \mathrm{Deg}(M).$$

The following gives a basic comparison between any $\text{Deg}(\cdot)$ function and the Castelnuovo–Mumford index of regularity of a standard graded algebra.

Let A be a standard graded algebra over an infinite field k. For any function $\text{Deg}(\cdot)$, it holds

$$\text{reg}(A) < \text{Deg}(A).$$

This justifies the use of $\text{Deg}(\cdot)$ as a measurer of complexities, but given the inequality, why bother? We will keep this issue on the table.

A comparison in the opposite direction for a particular $\text{Deg}(\cdot)$ will be given soon.

ARE THERE OTHER DEGS?

The set of cohomological degrees is clearly a convex set. By setting

$$\mathrm{bdeg}(M) = \inf\{\mathrm{Deg}(M)\},$$

where $\mathrm{Deg}(\cdot)$ runs over the set of all the cohomological degrees, obviously one gets another degree. In his thesis, Tor Gunston found a way to refine any given cohomological degree function $\mathrm{Deg}(\cdot)$ into the unique cohomological degree $\mathrm{bdeg}(\cdot)$ that meets the Bertini's condition strictly: If M has positive depth there are generic hyperplane sections such that

$$\mathrm{bdeg}(M) = \mathrm{bdeg}(M/hM).$$

He accomplishes this by defining

$$\mathrm{bdeg}(M) := \max\{\ \mathrm{bdeg}(M/hM) \mid h \text{ is a generic hyperplane section }\}.$$

It takes however an intricate technical argument to establish the coherence of this definition. One drawback is that even when $\operatorname{Deg}(\cdot)$ is given by an explicit formula–such as in the case of $\operatorname{hdeg}(\cdot)$–the determination of $\operatorname{bdeg}(M)$ does not come easy. The existence of at least one $\operatorname{Deg}(\cdot)$ is used in his argument to establish the existence of the maximum in the definition above.

Theorem[Gunston] If M is a finitely generated module over a homogeneous k–algebra and depth$M > 0$, then

$$\operatorname{bdeg}(M) \leq H(M;r),$$

where $H(M;t)$ is the Hilbert function of M and $r = \operatorname{reg}(M)$.

PROBLEMS

- For two finitely generated R–modules, estimate

$$\nu(\mathrm{Hom}_R(A, B))$$

- For an ideal $I \subset R$, figure out bounds for

$$\mathrm{Deg}(\mathrm{gr}_I(R))$$

- Interpret the (super) Hilbert function of the standard graded algebra $S = \oplus_{n \geq 0} S_n$ over the local ring R

$$SH(S; t) = H_{\mathrm{bdeg}}(S; t) = \sum_{n \geq 0} \mathrm{bdeg}(S_n) t^n$$

- Do we need Higher Degs?

$\mathrm{Hom}_R(A,B)$

The number of generators of $\mathrm{Hom}_R(A,B)$ obviously depends on many relationships between the modules A & B. For a limited cass of modules, we may want to frame the issue as follows: Consider the function $\mathrm{Hom}_R(A,\cdot)$ (or $\mathrm{Hom}_R(\cdot,B)$) and find estimates for the number of generators of its values. $\mathrm{Deg}(\cdot)$ functions seem to have a role that might lead to something like

$$\mathrm{Deg}(\mathrm{Hom}_R(A,B)) \leq f(\mathrm{Deg}(A),\mathrm{Deg}(B)),$$

where $f(\cdot,\cdot)$ is a polynomial (depending on R).

Partly, the motivation is to be able to find estimates for the number of generators of modules such as such as $[I \subset R]$

$$\mathrm{Hom}_R(I,I) \quad \text{and} \quad \bigcup_{n\geq 1} \mathrm{Hom}_R(I^n,I^n)$$

Using $\text{Deg}(\cdot) = \text{hdeg}(\cdot)$ or $\text{bdeg}(\cdot)$, with Kia Dalili, we have some situations where

$$\text{hdeg}(\text{Hom}_R(A,B)) \leq f(\text{hdeg}(A) \cdot \text{hdeg}(B))$$

where $f(\cdot)$ is a low degree polynomial (3 when $\dim R = 2$).

The restrictions arise as follows: If R is an affine domain over an infinite field, consider a hypersurface ring $S \subset R$ (finite+rational). For any two free modules A, B, $\text{Hom}_R(A,B) = \text{Hom}_S(A,B)$, so the restriction '$R$ is Gorenstein' is not very damaging. Another restriction is that we are mostly interested in $\text{Hom}_R(E,E)$, E^{**} and E^{*}.

Theorem: If E is a module of dimension $d \geq 1$ over a Gorenstein ring R, then

$$\text{bdeg}(E^{*}) \leq (\deg(R) + \binom{d}{2})\text{hdeg}(E).$$

Example: Assume $\dim R = 3$, E is a torsionfree module of projective dimension 1,

$$0 \to F_1 \longrightarrow F_0 \longrightarrow E \to 0,$$

and that we want $\nu(\mathrm{Hom}(E,E))$. The setting is the usual [Auslander-Bridger]

$$E^* \otimes E \longrightarrow \mathrm{Hom}(E,E) \longrightarrow \mathrm{Tor}_1(D(E),E) \to 0,$$

where $D(E)$ in this case is $\mathrm{Ext}^1(E,R)$. (Note that "we know a lot" about this module since it appears in $\mathrm{hdeg}(E)$.) Looking at the sequence

$$0 \to \mathrm{Tor}_1(E,D(E)) \to F_1 \otimes D(E) \to F_0 \otimes D(E) \to E \otimes D(E) \to 0,$$

and since $D(E)$ has dimension at most 1,

$$\mathrm{Deg}(\mathrm{Tor}_1(E,D(E)) \leq \mathrm{Deg}(F_1 \otimes D(E)) = f_1 \cdot \mathrm{Deg}(D(E)) \leq f_1 \cdot (\mathrm{hdeg}(E) - \deg(E))$$

$f_1 = \beta_1(E)$ is the Betti number of E which we know about from its $\mathrm{hdeg}(E)$ (and the ring R: $\beta_1(E) \leq \beta_1(K)\mathrm{Deg}(E)$, and make use of a previous general result for $\nu(E^*)$). Putting together

$$\nu(\mathrm{Hom}(E,E)) \leq (\deg(R)+1)\mathrm{hdeg}(E)\nu(E) + f_1(\mathrm{hdeg}(E) - \deg(E))$$

Proposition: Let R be a regular local ring of dimension 3 and let A and B be reflexive modules. Then

$$\text{hdeg}(\text{Hom}_R(A,B)) \leq 2 \cdot \text{hdeg}(A) \cdot \text{hdeg}(B).$$

Proof. Since A is reflexive, it has a projective resolution

$$0 \to R^m \longrightarrow R^n \longrightarrow A \to 0;$$

a simple calculation shows that

$$\text{hdeg}(A) = \deg(A) + \ell(\text{Ext}_R^1(A,R)),$$

since $\text{Ext}_R^1(A,R)$ is a module of finite length. We use a similar expression for $\text{hdeg}(\text{Hom}_R(A,B))$.

Applying the functor $\text{Hom}_R(\cdot,B)$ to the resolution yields the exact sequence

$$0 \to H = \mathrm{Hom}_R(A,B) \longrightarrow B^n \xrightarrow{\varphi} B^m \longrightarrow \mathrm{Ext}^1_R(A,B) \to 0.$$

Denote by D the image of φ and consider the cohomology exact sequences (we take into account the fact that A, B and $\mathrm{Hom}_R(A,B)$ are reflexive modules and $\mathrm{Ext}^1_R(A,R)$, $\mathrm{Ext}^1_R(B,R)$ and $\mathrm{Ext}^1_R(A,B)$ are modules of finite length):

$$\begin{aligned} 0 \to \mathrm{Hom}_R(D,R) \to \mathrm{Hom}_R(B^n,R) \to \mathrm{Hom}_R(H,R) \to \mathrm{Ext}^1_R(D,R) \\ \to \mathrm{Ext}^1_R(B^n,R) \to \mathrm{Ext}^1_R(H,R) \to \mathrm{Ext}^2_R(D,R) \to 0, \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_R(B^m,R) &\simeq \mathrm{Hom}_R(D,R) \\ \mathrm{Ext}^1_R(B^m,R) &\simeq \mathrm{Ext}^1_R(D,R) \\ \mathrm{Ext}^2_R(D,R) &\simeq \mathrm{Ext}^3_R(\mathrm{Ext}^1_R(A,B),R). \end{aligned}$$

With these identifications, and the exact sequence

$$\mathrm{Ext}^1_R(B,R^m) = \mathrm{Ext}^1_R(B^m,R) \longrightarrow \mathrm{Ext}^1_R(B^n,R) = \mathrm{Ext}^1_R(B,R^n) \longrightarrow \mathrm{Ext}^1_R(B,A) \to 0,$$

we obtain the exact sequence

$$0 \to \mathrm{Ext}^1_R(B,A) \longrightarrow \mathrm{Ext}^1_R(\mathrm{Hom}_R(A,B),R) \longrightarrow \mathrm{Ext}^3_R(\mathrm{Ext}^1_R(A,B),R) \to 0.$$

Next observe that $\mathrm{Ext}^1_R(B,A) \simeq \mathrm{Ext}^1_R(B,R) \otimes_R A$, and therefore

$$\ell(\mathrm{Ext}^1_R(B,A)) \leq \nu(A)\ell(\mathrm{Ext}^1_R(B,A)).$$

By duality we have a similar expression for the length of $\mathrm{Ext}_R^3(\mathrm{Ext}_R^1(A,B),R)$. Assembling $\mathrm{hdeg}(\mathrm{Hom}_R(A,B))$ we have

$$\begin{aligned}
\mathrm{hdeg}(\mathrm{Hom}_R(A,B)) &= \deg(\mathrm{Hom}_R(A,B)) + \ell(\mathrm{Ext}_R^1(\mathrm{Hom}_R(A,B),R)) \\
&\leq \deg(A)\cdot\deg(B) + \nu(A)(\mathrm{hdeg}(B) - \deg(B)) \\
&+ \nu(B)(\mathrm{hdeg}(A) - \deg(A)) \\
&\leq 2\cdot\mathrm{hdeg}(A)\cdot\mathrm{hdeg}(B),
\end{aligned}$$

since $\nu(\cdot) \leq \mathrm{hdeg}(\cdot)$.

Exercise with \$10 reward: Let R be a Cohen–Macaulay local ring with canonical module ω and define the module E by

$$0 \to R \longrightarrow \omega^{\oplus r} \longrightarrow E \to 0,$$

where $1 \in R$ maps onto the 'vector' $(x_1, \ldots, x_r)$ determined by a generating set for ω. E is Cohen–Macaulay. What is a bound for

$$\nu(\mathrm{Hom}_R(E,E))?$$

$\mathrm{Deg}(\mathrm{gr}_I(R))$

One setting for applications of $\mathrm{Deg}(\cdot)$ to finiteness results is the following elementary observation:

Proposition: Let $\mathrm{Deg}(\cdot)$ be a cohomological degree definable on local Noetherian rings. Given two positive integers A and d, there exists only a finite number of Hilbert functions associated to standard graded algebras G over Artinian rings such that $\dim G = d$ and $\mathrm{Deg}(G) \leq A$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and set $G = \mathrm{gr}_{\mathfrak{m}}(R)$.

Rossi, Trung and Valla established the following elegant count:

Theorem: For any cohomological degree $\mathrm{Deg}(\cdot)$ set
$I(R) = \mathrm{Deg}(R) - \deg(R) = \mathrm{Deg}(R) - e(R)$. The following estimation holds

$$\mathrm{reg}(G) \leq \begin{cases} \mathrm{Deg}(R) - 1 & \text{if } d = 1, \\ e(R)^{(d-1)!-1}[e(R)^2 + e(R)I(R) + 2I(R) - e(R)]^{(d-1)!} - I(R) & \text{if } d \geq 2. \end{cases}$$

Conjecture: Let I be an ideal of a Cohen–Macaulay local ring R. Then

$$\mathrm{r}(I) < \mathrm{Deg}(\mathrm{gr}_I(R)),$$

where $\mathrm{r}(I)$ is the reduction number of I.

$SH(S;t)$

Let $(R,\mathfrak{m})$ be a Noetherian local ring with an infinite residue field and fix $\mathrm{Deg}(\cdot) = \mathrm{bdeg}(\cdot)$. For a (an essentially) standard graded algebra S over R,

$$S = \bigoplus_{n \geq 0} S_n,$$

$$SH(S;t) = \bigoplus_{n \geq 0} \mathrm{Deg}(S_n) t^n,$$

which will be called the Big Hilbert Series of S. (Similarly for graded modules.)

Proposition: $SH(S;t)$ is a rational function of degree $\dim S/\mathfrak{m}S$.

If $I \subset R$ and $S = \mathrm{gr}_I(R)$, the knowledge of such functions would be helpful in studying properties of the integral closure of I (reduction numbers, multiplicity of fibers particularly) in the same manner one already has for $\mathfrak{m}$–primary ideals.

Drawback: Pretty tricky to compute.

What is the meaning of the **postulation number** of $SH(S;t)$?

Are Higher Degs needed?

- Computation Driven: Combining $\mathrm{reg}(\cdot)$ with $\mathrm{Deg}(\cdot)$?
- Digging Deeper: Using Hilbert function more?