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"Intersection homology D-module in pos. char."

$$Y \overset{c}{\subseteq} X$$

X: dim n, smooth$/k$

constructible $\mathbb{C}$ - $\forall s$

(holonomic D-modules)

$$\begin{array}{ccc} & & \mathcal{H}^c_Y(X, \mathcal{O}_X) \\ & \overset{RH}{\longleftrightarrow} & \cup\!\!| \\ IC^\cdot(Y) & \longleftrightarrow & \mathcal{L}(Y, X) \end{array}$$

$\mathcal{H}^c_Y(X, \mathcal{O}_X)$ ← D_X - module

$\mathcal{L}(Y, X)$ ← unique simple D_X - submodule

$$\mathcal{H}^c_Y(X, \mathcal{O}_X)\Big|_{X - \operatorname{Sing} Y} = \mathcal{L}(Y, X)\Big|_{X - \operatorname{Sing} Y}$$

<u>Thm. 1</u> $Y \subseteq X$ as before (char (k) arbitrary)

$\exists!$ simple D_X - submodule

$$\mathcal{L}(Y, X) \overset{*}{\subseteq} \mathcal{H}^c_Y(X, \mathcal{O}_X)$$

finite length (in char. p Lyubeznik)

Moreover, away from $\operatorname{Sing} Y$ $*$ is "="

(A) proof

(B) construction of $\mathcal{L}$

(C) Corollaries

(D) R-H like correspondence

(A) $i: X - Z \subseteq X$, $Z = \operatorname{sing} Y$

$$\mathcal{L} \subseteq \mathcal{H}^c_Y(X, \mathcal{O}_X) \subseteq i_* \mathcal{H}_{X-Z}(X-Z, \mathcal{O}_{X-Z})$$

any simple $\mathcal{D}_X$-submod.

$$\mathcal{H}_{Y-Z}(X-Z, \mathcal{O}_{X-Z}) \leftarrow \text{simple}$$

$$\mathcal{L}|_{X-Z} \subseteq \mathcal{H}^c_Y(X, \mathcal{O}_X)|_{X-Z} \subseteq i_* \mathcal{H}_{X-Z}(X-Z, \mathcal{O}_{X-Z})|_{X-Z}$$

$$\mathcal{L}|_{X-Z} \neq 0$$

$$\operatorname{Hom}(\mathcal{L}, i_* \mathcal{H}_{X-Z}) = \operatorname{Hom}(\mathcal{L}|_{X-Z}, \mathcal{H}_{X-Z})$$

$$\operatorname{Hom}(\mathcal{L}, i_* \mathcal{H}_{X-Z}) \neq 0$$

$Y \subseteq X$ both smooth

$$\hookrightarrow \mathcal{H}_Y(X, \mathcal{O}_X)$$

$\mathcal{O}_Y$ simple

replace D-module structure by Frobenius : (char $= p > 0$)

unit $R[F]$-modules :

$\left\{ \begin{array}{l} X = \operatorname{Spec}(R) \leftarrow \text{regular} \\ Y = \operatorname{Spec}(\underbrace{R/I}_{(A,m)}) \\ (R,m) \text{ is local.} \end{array} \right.$

Def. $R[F]$ module is an R-module M & R-lin. map
$\vartheta : F^*M \to M$

$F : R \to R$
$\quad r \mapsto r^p$

$$\begin{array}{c} F:\ \operatorname{Spec} R \\ \downarrow \\ \operatorname{Spec} R \end{array}$$

$R^{(e)}$: R-R-bimodule

left action : usual
right action : via F
$m \cdot r = r^{p^e} m$

note: $F^*M = R^{(1)} \otimes M$

(M, ϑ) is called unit if ϑ is "$\cong$".

Rmk. $R[F] := \dfrac{R\langle F \rangle}{\langle r^p F - Fr \mid r \in R \rangle}$

$\vartheta \in \operatorname{Hom}_R(F^*M, M) \cong \operatorname{Hom}_R(M, F_*M)$

$\longrightarrow F_\vartheta(rm) = r^p F_\vartheta(m)$

<u>examples</u> • R is unit $R[F]$-module:

$$\begin{aligned} R^{(1)} \otimes R &\longrightarrow R \\ r \otimes s &\longmapsto r s^p \\ r \otimes 1 &\longleftarrow r \end{aligned}$$

• $f \in R$ $\qquad R_f$ is unit $R[F]$-module.

$$\begin{aligned} R^{(1)} \otimes R_f &\longrightarrow R_f \\ r \otimes t/s &\longmapsto r \frac{t^p}{s^p} \\ t s^{p-1} \otimes \frac{1}{s} &\longleftarrow t/s \end{aligned}$$

R_f is gen. by $\frac{1}{f}$ as $R[F]$-module.

<u>Thm.</u> (R regular) Cat f.g. unit $R[F]$-modules is
(1) abelian and closed under extensions
(2) all f.g. u. $R[F]$-modules have finite length as such.

<u>Def.</u> $$D_R := \bigcup_e \operatorname{Hom}_{R^{p^e}}(R, R)$$
$$= \bigcup_e \operatorname{Hom}_{\text{mod-}R}(R^{(e)}, R^{(e)})$$

note: $\frac{\partial}{\partial x} : R \to R$ sends $R^p \to 0$.

$\frac{\partial^p}{p!\,\partial x^p}$ lin. over R^{p^2}

Thm. (cont'd)

(3) also finite length as a D_R-module

(length D_R = length u. $R[F]$, $k=\bar{k}$)

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setup:

$$\begin{tikzcd}
\mathcal{H}^c_I(R) \arrow[r, leftrightarrow] & H^{d}_{m}(A) \arrow[dl] & \supseteq 0^* \\
\mathcal{L}(A,R) \arrow[u, phantom, "\cup" description] \arrow[r, leftrightarrow] & H^d_m(A)/0^* &
\end{tikzcd}$$

$d \to \dim A$

$0^* = \{\eta \mid \exists c \neq 0 \ c\eta^{p^e} = 0 \ \forall e\}$

max'l proper $R[F]$-submod. since A domain, irred.

$H^d_m(A)/0^*$ simple $R[F]$-submod.

Thm. $\mathcal{L}(A,R) = \left(H^d_m(A)/0^*\right)^{\vee_F}$

Cor. • if A is F-rational, i.e. $0^* = 0$, then $H^c_I(R)$ is D-simple.

• $H^c_I(R)$ is simple $\Longleftrightarrow$ 0^* is F-nilpotent

$$(A \text{ C-M}) \iff (x_1, \dots, x_d)^* = (x_1, \dots, x_d)^F$$

for any s.o.p.

<u>Cor.</u> ~~A curve~~ Y curve, $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ is simple

$\iff$ $\tilde{Y} \to Y$ is injective.

<u>example</u> $A = k[x, y, z, w] / \underbrace{xy - zw}_{=f}$

$\xRightarrow{\text{char } p}$ $H^1_f(R)$ simple

$\xRightarrow{\text{char } 0}$ $b_f = (s+1)(s+2)$