Hubert Flenner Dec. 4, '02

"$\mathbb{C}^*$- and $\mathbb{C}_+$-actions on affine surfaces."

joint with M. Zaidenberg

Affine surfaces / $\mathbb{C}$: $X = \operatorname{Spec} A$ normal

Orlik, Wagreich, Fieseler–Kaup, ...

part I : $\mathbb{C}^*$-actions on such surfaces

II : $\mathbb{C}_+$ & $\mathbb{C}_*$-actions on such surfaces

part I $\mathbb{C}^*$-actions on $X = \operatorname{spec} A$

$$\Rightarrow \quad A = \bigoplus_{i \in \mathbb{Z}} A_i \ , \quad A_i = \{ f \in A \mid t.f = t^i f \ \forall t \in \mathbb{C}^* \}$$

3 cases (Fieseler – Kaup)

elliptic	parabolic	hyperbolic
$A_0 = \mathbb{C}$	$A_0 =$ 1-dim. domain	$A_- \neq 0$, $A_+ \neq 0$
$A_- = \bigoplus_{i<0} A_i = 0$	$A_- = 0$	
	$\pi : X \to C = \operatorname{spec}(A_0)$	$\pi : X \to C$
	$\mathbb{C}_+$ - filtration	$\mathbb{C}^*$-filtration

elliptic case: Pinkham – Demazure – Dolgachev :

C = proj. smooth curve , $D \in \mathbb{Q}\mathrm{Div}(X)$, $K = \mathrm{Frac}(C)$

$A = A_{C,D} \subseteq K[u]$

$A_n := H^0(\mathcal{O}_C(\lfloor nD \rfloor))\, u^n \subseteq K u^n$

$H^0(\mathcal{O}_C(\lfloor nD \rfloor)) = \{ f \in K \mid \mathrm{div}\, f + nD \geq 0 \}$

Thm E^* 1) every elliptic $X = \mathrm{Spec}(A)$ arises in this way.

2) $A_{C,D} \cong A_{C',D'} \Leftrightarrow C \cong C'$, $D = D' + \mathrm{div}\, f \quad f \in K$

parabolic case Demazure:

Let C = affine smooth curve $= \mathrm{Spec}(A_0)$,

$D \in \mathbb{Q}\,\mathrm{Div}(C)$.

$A = A_0[D] \subseteq K[u]$, where $A_n = H^0(\mathcal{O}_C(\lfloor nD \rfloor))\, u^n$

Thm P^* 1) every parabolic A arises in this way

2) $A_0[D] \cong A_0[D'] \Leftrightarrow D = D' + \mathrm{div}(f) \quad f \in K$

Hyperbolic case

basic observation: $A_{\geq 0} = \bigoplus_{i \geq 0} A_i = A_0 [D_+]$

$$A_{\leq 0} = \bigoplus_{i \leq 0} A_i = A_0 [D_-]$$

$A = A_0 [D_+, D_-]$.

in general: $C = \operatorname{Spec} A_0$ affine smooth curve

$D_\pm \in \mathbb{Q}\operatorname{Div}(C)$; $A := A_0[D_+, D_-] \subseteq K[u, u^{-1}]$,

where
$$A_n := \begin{cases} H^0(\mathcal{O}_C(\lfloor n D_+ \rfloor))\, u^n & n \geq 0 \\ H^0(\mathcal{O}_C(\lfloor -n D_- \rfloor))\, u^n & n < 0 \end{cases}$$

observation: $A \subseteq K[u, u^{-1}]$ subring $\Leftrightarrow$ $D_+ + D_- \leq 0$.

Thm. H^* 1) every hyperbolic 2-dim normal f.g. $\mathbb{C}$-alg. A is isom. to $A_0[D_+, D_-]$ as before.

2) $A_0[D_+, D_-] \cong A_0[D_+', D_-'] \Leftrightarrow$

$\exists f \in K$ s.t. $D_+ = D_+' + \operatorname{div}(f)$,
$D_- = D_-' - \operatorname{div}(f)$.

examples (1) parabolic: A = normal. of

$$B = A_0[u,v]/(u^d - vP), \qquad P \in A_0$$

graded via $\deg u = 1$, $\deg v = d$.

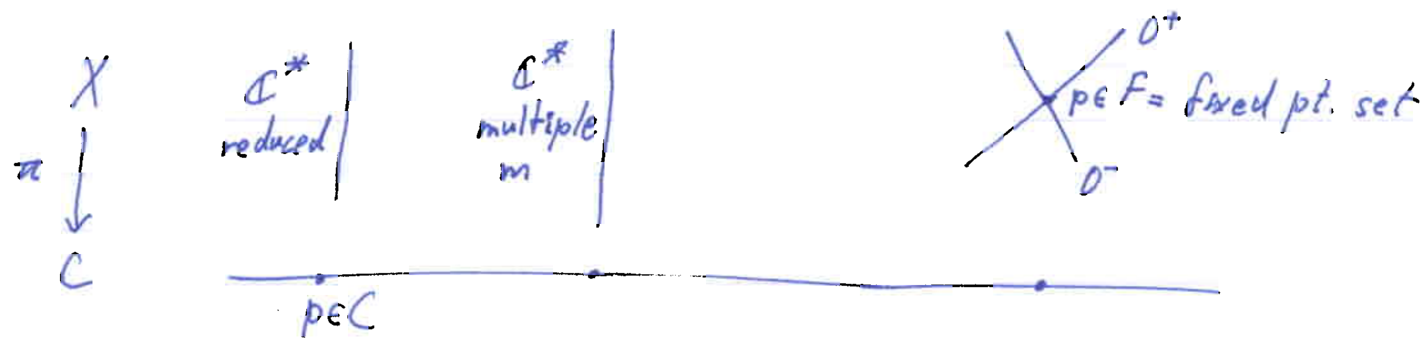
$$A = A_0\left[\frac{\operatorname{div} P}{d}\right].$$

(2) hyperbolic A = normal. of $B = A_0[u,v]/(u^d v - P)$

$\deg u = 1$, $\deg v = -d$

$$A = A_0\left[0, -\frac{\operatorname{div} P}{d}\right].$$

Applications 1) smoothness of $\operatorname{Spec}(A) = X$



$D_\pm(p) = 0$

$D_+(p) + D_-(p) = 0$

$D_\pm(p) \neq 0$

$D_+(p) = -\frac{e}{m}$

$D_+(p) + D_-(p) < 0$

$\operatorname{Sing}(X) \subseteq F$

$D_+(p) = -\frac{e_+}{m_+}, \quad D_-(p) = \frac{e_-}{m_-}$

$$p \in \operatorname{Sing}(X) \iff \Delta(p) = -\begin{vmatrix} e_+ & e_- \\ m_+ & m_- \end{vmatrix} \neq 1$$

2) Can calculate $Cl(A)$, $Pic(A)$, ω_A

Part II $\mathbb{C}_+$-actions

A = f.g. $\mathbb{C}$-alg.

Def. $\partial : A \to A$ derivation is called locally nilpotent (LN) if $\forall a \in A\ \exists n = n(a) : \partial^n(a) = 0$

example: $\mathbb{C}[x_1, \dots, x_n]$, $\frac{\partial}{\partial x_i}$ are LN.

Can &o ∂ a $\mathbb{C}_+$-action (Reubchler)

$t \in \mathbb{C}^*$, $f \in A$ then $t.f := \sum_{n=0}^{\infty} \frac{t^n \partial^n(f)}{n!}$

Thm. (Reubchler) 1-1 correspond. between $\mathbb{C}_+$-actions and LN-derivations.

$A = \bigoplus_{i \in \mathbb{Z}} A_i$ graded

Prop. 1) If $\partial : A \to A$ LN, then ∂ is homog.

$\Leftrightarrow$ $\exists$ action of $\mathbb{C}^* \ltimes \mathbb{C}_+ = G$ on $X = \operatorname{Spec} A$

2) if A admits a LN derivation $\Longrightarrow$ also admits a homog. one.

In the following we classify (A, ∂),

A = 2-dim normal graded f.g. $\mathbb{C}$-alg.,

∂ = <u>hom</u> LN derivation.

<u>Thm. E_+</u> If (A, ∂) as above is elliptic, then

$A = \mathbb{C}[X, Y]^{\mathbb{Z}_d}$, $\quad \mathbb{Z}_d = \langle \zeta \rangle$

ζ = prim d-th root of 1

$\zeta . X = \zeta X$
$\zeta . Y = \zeta^e Y$ and $\partial = \tilde{\partial}|_A$, $\tilde{\partial}$ on $\mathbb{C}[X,Y]$

$\tilde{\partial}(X) = 0$, $\tilde{\partial}(Y) = X^e$

$e := \deg \partial$

<u>Corollary</u> (J. Wahl)

If A has der. of neg. degree ($\Rightarrow$ LN)

then A = cyclic quotient.

<u>Parabolic case</u> (A, ∂) parabolic, ∂ as before.

Two different cases:

- (A, ∂) of fibre type, if the $\mathbb{C}_+$-action preserves fibres of π.
- (A, ∂) horizontal otherwise

  $(\Rightarrow C = \mathbb{A}^1, \quad A_0 = \mathbb{C}[z])$

<u>Thm. P_+</u> (a) If fibre type, then

$$\partial \overset{1-1}{\longleftrightarrow} \varphi \in H^0(\mathcal{O}_C[-D]), \quad \deg \partial = -1$$

(b) If (A, ∂) horizontal, then, with $e := \deg \partial$,

$A = \mathbb{C}[s, u']^{\mathbb{Z}_d}, \quad \mathbb{Z}_d = \langle \zeta \rangle,$

$\deg s = 0, \ \deg u' = 1.$

$$\zeta \cdot s = \zeta s$$

$$\zeta \cdot u' = \zeta^{e'} u' \qquad ee' \equiv 1 \ (d), \quad 0 \le e' \le d$$

$$\partial = u'^{e} \frac{\partial}{\partial s}.$$

Hyperbolic case $(A, \partial) =$ hyperbolic, $e = \deg \partial$.

May assume: $e \geq 0$! $\Rightarrow \partial(A_{\geq 0}) \subseteq A_{\geq 0}$

thus we know ∂ on $\operatorname{Frac}(A)$.

Thm M_+

$d := \min\{n \mid n D_+ = \text{integral}\}$

where $A = A_0[D_+, D_-]$

$A = A_0[D_+, D_-] \subseteq K[u, u^{-1}]$

with: 1) $A_0 = \mathbb{C}[t]$

2) $D_+ = -\frac{e'}{d}[0]$ where $ee' \equiv 1\ (d)$, $0 \leq e' < d$

3) if $p \neq 0$ and $D_-(p) \neq 0 \Rightarrow e D_-(a) \leq -1$

$p = 0$ and $D_-(0) = \frac{e'}{d} \Rightarrow e D_-(0) \leq \frac{ee'-1}{d}$

4) $\partial = t^k u^e \left(dt \frac{\partial}{\partial t} - e' u \frac{\partial}{\partial u} \right)$, $k = \frac{ee'-1}{d}$