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"Extended plus closure and colon-capturing"

R local Noetherian domain of mixed char.

R^+ = integral closure of R in alg. closure of quotient field.

Def. I ideal of R, $x \in R$

(a) $x \in I^+$, plus closure, if $x \in IR^+$

(b) $x \in I^{epf}$, full extended plus closure, if

$\exists\ c \neq 0 \in R$ s/t $c^{1/N} x \in (I, p^N)R^+$

$\forall\ N \in \mathbb{Z}^+$

Direct summand conjecture: (D.S.C.)

Let R be a RLR, $R \subseteq S$ module finite extension

$\exists$ splitting map $S \to R$

Hochster: D.S.C. holds if $I^+ = I$

whenever R is regular

this is equivalent in fixed dimension.

Def. We say a closure operation is colon-capturing if whenever $x_1, \ldots, x_n$ are parameters, $(x_1, \ldots, x_{n-1}) :_R x_n \subseteq (x_1, \ldots, x_{n-1})^{\text{closure}}$

Thm. 1 If full extended plus closure has colon capturing property, for complete local domains, $I^{epf} = I$ whenever R is regular.

Thm. 2 Assume p, x, y are parameters, $\exists\, \sigma \in R$ with $\sigma^{p-1} = p$, $p^k z \in (x, y)R$, p^k, x, y kills $H_1(x^m, y^m, p^m; R)$ $\forall m$.

Then, for any $N \in \mathbb{Z}^+$ $p^{1/N} z \in (x, y) R^+$.

Equivalently, $\exists$ module finite extension S with $p^{1/N} z \in (x, y) S$.

Thm 3. (Faltings) If R is excellent and normal and x, y, z are parameters, then $\exists\ k$ s/t x^k, y^k, z^k kill $H_1(x^m, y^m, z^m; R)$ $\forall\ m$.

Thm. 4 Let R be an excellent normal domain. Suppose p, x, y are parameters. Assume $p^k z \in (x,y)R$

(a) for any N, $p^{1/N} z \in (x,y)R^+$

(b) if in addition, R is graded and x, y, z are homogeneous, we may construct a module finite $S = R[p^{1/N}, \alpha, \beta]$ s.t. $p^{1/N} z = x\alpha + y\beta$ and α, β are homog.

Pf (a): follows from Thm 2, 3.

Pf (b): In (a), we find $p^{1/N} z = x\alpha^* + y\beta^*$.

To prove Thm. 2, we construct a polynomial

$f(T) = T^n + a_1 T^{n-1} + \dots + a_n$ and

let α^* be a root.

$$\beta^* = \frac{p^{1/n} z - \alpha^* x}{y}.$$ It's polynomial;

can be found by Taylor's Theorem.

Necessary condition for β^* integral is

$$\forall i = 1..n \quad \sum_{j=0}^{i} \binom{n-j}{i-j} a_j z^{i-j} x^j \in y^i R$$

Look at term of degree i ($\deg z$)

and let $\tilde{a}_j$ = degree j $\left(\deg\left(\frac{z}{x}\right)\right)$

$$\sum_{j=0}^{i} \binom{n-j}{i-j} z^i \tilde{a}_j \left(\frac{x}{z}\right)^j \in y^i R$$

$\tilde{a}_0 = 1 = a_0$

α root of $T^n + \tilde{a}_1 T^{n-1} + \dots + \tilde{a}_n$

<u>Lemma 5</u> Let R be excellent and normal.

Suppose x, y, z are parameters s.t. $zw \in (x, y)R$.

(*) Further suppose $ht(x, y, z, p) = 4$.

Then $w \in (x,y)^{epf}$. More specifically

$(pz)^{1/N} w \in (x, y, p^N) R^+ \quad \forall N$.

Proof: let $S = R[u, p^{N^2}t, zt]$ be the extended Rees algebra $t = u^{-1}$

(claim 1) $ht(P, x, y) S = 3$

verification: suppose $(p, x, y) \in \mathcal{P}$ prime

If $u \notin P$, P extends to a prime

$PS[u^{-1}] = PR[u, u^{-1}]$ and $ht\, p = ht(P \cap R) \geq 3$.

If $u \in P$, $z \in P$ also, $(p, x, y, z, u) \subset P$

$ht\ P \cap R[u] \geq 5$.

$S \cong R[u][X_1, X_2]/Q$ ht 2 prime

$ht\ P \geq 5 - 2 = 3$

Claim: $\exists\, p^{1/N}, \alpha, \beta \in S^+$ with α, β homog.

$p^{1/N}(ztw) = x\alpha + y\beta$. Enough to show this locally.

Obvious if prime doesn't contain (p, x, y)

$$zw = xa + yb$$

$$p^{N^2} t z w = x\bar{a} + y\bar{b}$$

$$p^{N^2}(ztw) = x\bar{a} + y\bar{b}$$

If p is not a unit, apply Thm. 4b.

$$p^{1/N} z t w = x\alpha + y\beta$$

$$p^{1/N} \underset{\underset{p^{1/N} zw}{\shortparallel}}{ztwu} = x(u\alpha) + y(u\beta)$$

$\Rightarrow$ homog. of deg 0 in u's

$$\Rightarrow u\alpha, u\beta \in \overline{(p^{N^2}, z)}$$

$$= \overline{(p^N, z^{1/N})^N}$$

proof: By Briançon - Skoda thm.,

$$u\alpha, u\beta \in (p^N, z^{1/N})^{N-1} R^+ \subset (p^N, z^{\frac{N-1}{N}}) R^+$$

$$(pz)^{1/N} zw \in (x,y)(p^N, z) R^+$$

$$(pz)^{1/N} zw \in z(x,y) R^+ + p^N R^+ = z(Ax+By) + p^N C$$

Lemma 6 Weaken (*) to (**) $ht(x,y,p)=3$

$zw = ax+by$

$(x,y)R = \underbrace{Q_1 \cap \dots \cap Q_k}_{\text{isolated}} \cap \underbrace{Q_{k+1} \cap \dots}_{\substack{\text{embedded} \\ p^k \in \text{here}}} \cap \underbrace{Q_\ell \cap \dots}_{p^k \notin \text{here}}$

$\Downarrow$
w

pick $v \in$ here

$p^k vw = Ax+By$ $\qquad$ $ht(x,y,p,v)=4$

$v(p^{1/N}w) \in (x,y)R^+$, finish by Lemma 5.

Thm. 7 Drop (**) .

$zw = ax+by$ $\qquad$ change x to $x+ry$

Assume $ht(P,x,z)=3$ $\qquad$ z to $z+s_1y+s_2x$

Gives $b \in (z,w)^{epf}$

$zw = ax+by$ $\rightsquigarrow$

$zw = ax+(A+Bz+Cp^N)y$ $\rightsquigarrow$

$z(w-By) = ax+\bar{C}p^N$