Anurag K. Singh Dec. 4, '02

"Direct Summands of module finite extension"

<u>Def.</u> (Ma, 1988) An integral domain R is a splinter if for every module finite extension ring S, the inclusion $R \hookrightarrow S$ splits via an R-lin. map.

<u>Rmk.</u> $R \subseteq S$ module-finite ring extension

① if $R \subseteq S$ splits, then $IS \cap R = I$ $\forall$ ideals $I \subseteq R$.

② if R is excellent, then $R \subseteq S$ splits $\Leftrightarrow$ for all ideals $I \subseteq R$ we have $IS \cap R = I$

splinter = "ideally integrally closed"

↳ Hochster, 1973 (Nagoya)

Assume rings are excellent, images of regular rings...

(3) splinter $\Longrightarrow$ normal

Pf. if $\frac{a}{b}$ is integral over R, then

$R \subseteq R[\frac{a}{b}]$ splits.

$$a = b\left(\frac{a}{b}\right) \in bR[\tfrac{a}{b}] \cap R = bR \quad \checkmark$$

(4) if $\mathbb{Q} \subseteq R$, then R is a splinter $\Longleftrightarrow$

R is normal.

Pf. if $R \subseteq S$ is module-finite domain

extension, then

$$\frac{1}{\mathcal{Q}(S):\mathcal{Q}(R)} \cdot \mathrm{Trace}_{\mathcal{Q}(S)/\mathcal{Q}(R)} : S \longrightarrow R$$

gives a splitting.

(5) Direct summand <u>conjecture</u>: "Every RLR

is a splinter."

open for rings of mixed char., $\dim \geq 4$.

(6) if $R = \mathbb{Z}_p[x,y,z]/(x^3+y^3+z^3)$, $p \neq 3$

This is normal.

$p=2$ $\quad z^2 \notin (x,y)R$

$$z^4 = zx^3 + zy^3$$

so $\quad z^2 = x\sqrt{zx} + y\sqrt{zy} \in (x,y)R^{1/2}$

$\Rightarrow$ R is not a splinter.

More generally, if $p \equiv 2 \bmod 3$, it is easy to check that $z^{2p} \in (x^p, y^p)R$

so $\quad z^2 \in (x,y)R^{1/p}$.

For $p \equiv 2 \bmod 3$, $R \subseteq R^{1/p}$ doesn't split

(supersingular elliptic curve)

If $p \equiv 1 \bmod 3$, then $R \subseteq R^{1/p}$ splits

(ordinary elliptic curve). However

$\exists$ separable extensions for which $R \subseteq S$

does not split.

R is not a splinter for every prime char. $p > 0$.

Let R be a domain of char $p > 0$, and S a module finite extension

$\exists$ R-hom. $\varphi: S \longrightarrow R$
$\qquad 1 \longmapsto c \neq 0$

Suppose $z \in IS \cap R$, $I \subseteq R$ ideal

then $\exists$ equation $z = \sum_{i=1}^{n} s_i x_i \qquad s_i \in S$, $x_i \in I$

$\forall q = p^e$ we have $z^q = \sum_i s_i^q x_i^q$

Apply φ : $cz^q = \sum \varphi(s_i^q)\, x_i^q$

i.e. $cz^q \in (x_1^q, \ldots, x_n^q) \quad \forall\, q = p^e$

i.e. $z \in I^*$.

Prop. $R \subseteq S$ module-finite, char $p > 0$.
For all ideals $I \subseteq R$ we have

$I \subseteq IS \cap R \subseteq I^*$

If every ideal is tightly closed (such rings are called weakly F-reg.) then R is a splinter. (using excelency!)

examples of weakly F-reg. rings (hence of splinters)

① Regular rings

② Determinantal rings

③ Plücker embedded Grassmanians

④ normal affine semigroup rings.

Let R^+ denote the integral closure of R in an alg. closure of its fraction field.

Thm. (Hochster - Huneke 1992) If R is a local domain of char $p > 0$, then R^+ is a big CM algebra for R.

If R is graded, R^{+gr} is a big CM algebra.

(—, 1999) R^{+sep} is a big CM algebra for R

$R^{+gr. sep}$ is not a big CM algebra.

related: R is a splinter $\Leftrightarrow$

R splits from separable module-finite extensions

Define $I^+ = IR^+ \cap R$. Then $I^+ \subseteq I^*$

Question: Is it true that $I^+ = I^*$?

— Yes, Smith (1994) if I is gen. by part of a system of parameters.

<u>Question</u> If a ring R has $I^+ = I \ \forall I$,

is it true that $I^* = I \ \forall I$?

i.e. does splinter $\Rightarrow$ weakly F-reg. ?

(note: we saw weakly F-reg. $\Rightarrow$ splinter)

splinter $\xrightarrow[\text{Smith}]{\text{Hochster, Huneke}}$ F-rational,
pseudo-rational,
C-M,
normal

1994, Hochster-Huneke: A Gorenstein splinter is weakly F-reg.

(1998, –) $\mathbb{Q}$-Gorenstein splinter $\Rightarrow$

weakly F-reg.

R is $\mathbb{Q}$-Gorenstein if $[\omega_R] \in Cl(R)$ has finite order.

Thm. ( ) If the anti-canonical cover of R is Noeth., then R splinter $\Rightarrow$ weakly F-reg.

Let $\mathfrak{a}$ be a divisorial ideal of R with $[\mathfrak{a}] = -[\omega_R]$ in $Cl(R)$.

The anti-canonical cover is

$$\mathcal{R}_s(\mathfrak{a}) = R \oplus \mathfrak{a} \oplus \mathfrak{a}^{(2)} \oplus \cdots$$

If R is Gorenstein, $\mathcal{R}_s(\mathfrak{a}) \simeq R[T] \Rightarrow$ Noeth.

If R is $\mathbb{Q}$-Gorenstein, $\mathcal{R}_s(\mathfrak{a})$ is Noeth.

Def. R is F-reg. if R_p is weakly F-reg. $\forall p$

Cor. Let R be a weakly F-reg. ring. If its anti-canonical cover is Noeth. on the punctured spectrum, then R is F-reg.

<u>example</u> (Mac Crimmon) If R is $\mathbb{Q}$-Gorenstein on the punctured spectrum, then

R is weak F-reg. $\Rightarrow$ R is F-reg.

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