CLOSURE OPERATIONS IN MIXED CHARACTERISTIC

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What is good about tight closure theory in equal characteristic? Under mild assumptions on the ring it

(1) Proves that direct summands of regular rings are Cohen-Macaulay

(2) Gives other local homological conjectures, e.g., it

(3) Proves that regular rings split from module-finite extensions

(4) Gives a generalized

Briançon-Skoda theorem

There are many other applications, including

(5) Gives comparison theorems for ordinary and symbolic powers of ideals in regular rings

But the definition is given in char. p, and the char. 0 definition depends on reducing to finitely generated algebras over the rationals and then to char. p.

We shall discuss several attempts to generalize tight closure to mixed char.: none has succeeded completely. Before discussing these closures, we want to mention some very important properties of tight closure. For simplicity, and because this case suffices for most of the applications, we focus throughout on what to do for ideals in a complete local domain.

(1) $I \subseteq I^* = (I^*)^*$

(2) $I \subseteq J \Rightarrow I^* \subseteq J^*$

(3) $x_1, \ldots, x_n$ part of a s.o.p. $\Rightarrow (x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$ (colon capturing)

(4) If R is regular, every ideal is tightly closed

(5) S module-finite over R implies $IS \cap R \subseteq I^*$

We might also hope for:

(3°) $x_1, \ldots, x_n$ part of a s.o.p. $\Rightarrow (x_1, \ldots, x_k)^* : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$

(6) Given $I \subseteq R$ and $R \to S$, $I^*S \subseteq (IS)^*$ (persistence)

A finite R-free complex $0 \to G_n \to \cdots \to G_0 \to 0$ is said to satisfy *the standard conditions on rank and height* if, with α_i the matrix of $R^{b_i} = G_i \to G_{i-1} = R^{b_{i-1}}$, and r_i the determinantal rank of α_i, we have for $1 \leq i \leq n$ that $b_i = r_i + r_{i-1}$ and height $I_{r_i}(\alpha_i) \geq i$ ($+\infty$ is allowed, i.e., the ideal may be R). With "depth" replacing "height" this gives an acyclicity criterion. For tight closure, we have:

(7) If $G_\bullet$ satisfues the standard conditons on rank and height then for all $i \geq 1$, the cycles are in the tight closure of the boundaries in G_i

This is called the *phantom acyclicity criterion* and may be thought of as a generalization of colon capturing. None of the attempts to construct a theory with these properties in mixed characteristic has been successful. In some cases, properties fail. In others, the theory might work, but it is not known how to prove this.

We'll discuss:

(1) Solid closure

(2) Big C-M algebra closures

(3) Parasolid closure (H. Brenner)

(4) Parameter tight closure

(5) Diamond closure (J. Velez and H)

(6) (Full extended) plus and full rank one closure (R. Heitmann)

Let (R, m, K) be a complete local domain of Krull dim d. Then an R-algebra (or module) S admits a nonzero R-linear map to R if and only if $H^d_m(S) \neq 0$. Such an algebra is called *solid*. A big C-M algebra S over R (one such that $mS \neq S$ and every s.o.p. for R is a regular sequence on S) is always solid. In char. $p > 0$ it turns out that $x \in I^*$ iff $x \in IS$ where S is a solid

R-algebra. Also, $x \in I^*$ iff $x \in IS$ where S is a big C-M algebra! Any condition in between "solid" and "big C-M" also characterizes tight closure in char. $p > 0$. Making these characterizations into definitions gives closure operations in all characteristics. Thus, one has *solid closure* and *big C-M closure*. I conjecture that there is a good extension of tight closure to mixed char.

iff big C-M algebras exist in mixed char. I also conjecture that the big C-M algebra notion will give a good notion if any good notion exists. However, it does not seem valuable right now, because in mixed char. we do not know that big C-M algebras exist.

The solid closure, which is bigger, has turned out to be useful, but there are difficulties. In studying solid closure there is a "semi-universal algebra" S one may try to use to force an element u into $(u_1, \ldots, u_n)S$: the *generic forcing algebra*. Let $\underline{X} = X_1, \ldots, X_n$ be variables and $S = R[\underline{X}]/(u - \sum_{i=1}^{n} X_i u_i)$. If there is any solid algebra T such that $u \in (u_1, \ldots, u_m)T$ then S works, so that $u \in \left((f_1, \ldots, f_n)R\right)^*$ iff $H^d_m(S) \neq 0$ for this choice of S. Starting here, H. Brenner, has shown that tight closure agrees with plus closure (soon to be described) for homogeneous m-primary ideals in homogeneous coordinate rings of elliptic curves in positive char. $p > 0$. His method brings the theory of vector bundles over elliptic curves to bear.

However, in equal characteristic 0, solid closure is definitely too big: a result of Paul Roberts shows that in

$R = \mathbb{Q}[[x, y, z]]$, where x, y, z are formal indeterminates, $x^2y^2z^2$ is in the solid closure of $(x^3, y^3, z^3)R$. Thus, ideals in regular rings are not solidly closed in equal char. 0. So far as I know, it remains an open question whether ideals of regular rings are solidly closed in mixed char., but the example of Roberts tends to make one pessimistic. Of course, it is natural to try to "correct" by giving a notion like solid closure in which solid algebras have been replaced by algebras satisfying a stronger condition.

H. Brenner has made an effort in this direction as follows. If R is a local domain of dimension d, the *paraclass* associated with the s.o.p. $x_1, \ldots, x_d$ for R is the class of $1/(x_1 \cdots x_d) \in H^d_m(S) = S_{x_1 \cdots x_d} / \sum_i S_{y_i}$, where $y_i =$

$x_1 \cdots x_{i-1}x_{i+1} \cdots x_n$. S is called a *parasolid* algebra if each paraclass that is non-zero in $H^d_m(R)$ (they all are nonzero in equal char.: in mixed char., this is equivalent to the direct summand conjecture) is non-zero in $H^d_m(S)$. S is called *universally parasolid* over R (which need not even be local) if for every map from R to a local ring R', $R' \otimes_R S$ is parasolid over R'.

The *parasolid closure* of I is the smallest ideal $J \supseteq I$ such that if $u \in JS$ in some universally parasolid algebra S over R, then $u \in J$. For this closure it is true that ideals of regular rings are closed. However, it is not known that a module-finite extension of a complete local regular ring is parasolid over it in mixed char.; which is problematic. Of course, a parasolid algebra over a complete local domain is solid, and so the parasolid closure is contained in the solid closure. Paul Roberts' example is not problematic in this theory, but other difficulties remain.

I have proposed a different modification of solid closure, called *parameter tight closure* which, again, is smaller than solid closure, but agrees with it in char. $p > 0$. One needs to define a *parameter-preserving* algebra S over a complete local domain R. The *parameter tight closure* of an ideal I is then the smallest ideal $J \supseteq I$ such that if S is parameter-preserving and $u \in JS$ then $u \in J$. S is *parameter-preserving* if every s.o.p. in R is *parameter-like* in S: this notion is defined by vanishing properties of local cohomology modules of some auxiliary algebras, as follows. Let $\mathcal{T}_0(S) = \mathcal{T}_0$ be the quotient of S by the ideal of all elements of R whose annihilator in R has positive height, and define $\mathcal{T}_i(S) = \mathcal{T}_i$ recursively by the rule that $\mathcal{T}_{i+1}$ is the quotient of $\mathcal{T}_i/x_{i+1}\mathcal{T}_i$ by the ideal consisting of all u such that $\dim Ru < d - i + 1$. Finally, $x_1, \ldots, x_d$ is *parameter-like* in S means that $\mathcal{T}_d \neq 0$ and for all i,

$0 \leq i \leq d-1$, the height of the annihilator of $H_m^{d-i-1}(\mathcal{T}_i)$ in R is at least $i+2$.

This is technical, but the condition implies that S is solid, holds for module-finite extension domains, and also holds for big Cohen-Macaulay algebras. It gives a notion that agrees with tight closure in char. $p > 0$. For this notion it is true that if S is module-finite over R, then $IS \cap R$ is contained in the parameter tight closure of I. However, we no longer know that ideals of regular rings are closed! On the third hand, it is at least true that the algebra Paul Roberts used to prove that ideals of regular rings are solidly closed is not parameter-preserving. It remains open whether ideals of regular rings are closed under parameter tight closure.

Write $q = p^e$. Let $I^{[q]} = (y^q : y \in I)R$: it is generated by q th powers of generators of I. The original definition of tight closure in a domain of char. $p > 0$ is that $u \in I^*$ if for some $c \neq 0$, $cu^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. Instead, one may write $u \in I^*$ if for some $c \neq 0$, $c^{1/q}u \in IR^{1/q}$ for all $q \gg 0$. However, weaker conditions suffice. Let R^+ denote the integral closure of the domain

R in an algebraic closure of its fraction field. It is unique up to non-canonical isomorphism. Write I^+ for $IR^+ \cap R$, the *plus closure* of I. Then $I \subseteq I^+ \subseteq I^*$, and it is possible that $I^+ = I^*$ always: Karen Smith proved this for ideals generated by part of a s.o.p. H & H proved that $u \in I^*$ iff there are elements $c_j \neq 0$ of R^+ of arbitrarily small order (with respect to some $\mathbb{Q}$-valued valuation positive on the maximal ideal of R) such that $c_j u \in IR^+$ for all j. This is quite a bit weaker than what is asserted in the definition of tight closure, where the c_j have the form $c^{1/q}$ for fixed c.

Both the original definition and this apparently weaker condition suggest various ways to extend the notion to mixed char. One idea, *diamond* *closure*, explored by J. Velez and H, is the following. In a complete local domain R of mixed char. p define $u \in I^{\diamondsuit}$ if for some c not in any minimal prime of p, $cu^n \in I^{\langle n \rangle}$ for all $n \gg 0$. Here, if $I = (f_1, \ldots, f_d)R$ and $n = qr$ where q is a power of p and p does not divide r, we denote by $I^{\langle n \rangle}$ the ideal of R generated by all elements of the form $q_1(f_1^{a_1} \cdots f_d^{a_d})^{q_2}$ where

$q = q_1q_2$ is a factorization of q into powers of p and the a_i are nonnegative integers whose sum is q_1r. It is not obvious, but it is true, that this is independent of the choice of generators for I. This gives a notion that coinicides with tight closure in good cases in char. $p > 0$. It yields colon-capturing results not obtainable by other means, and it is significantly smaller than integral closure. However, it is not true that ideals of regular local rings are closed, and it is clear that this is not the “ideal” generalization of tight closure to mixed char.

More promising is the characterization that says that $u \in I^*$ if there are non-zero elements of arbitrarily small order in R^+ that multiply u into IR^+. Very significant work in this area has been done by Ray Heitmann. Note that plus closure without "enhancement" won't work to capture colon ideals in mixed characteristic, because in dimension 4 or more R^+ is not a big C-M algebra in mixed char. So far as I know, it is an open question whether R^+ is a big Cohen-Macaulay algebra in mixed char. in dim. 3. However, Heitmann has proved the following remarkable result: if R is a complete local domain in mixed char. and x, y, z is a system of parameters, then for every $N > 0$, $p^{1/N}\big((x, y) :_R z\big) \subseteq (x, y)R^+$. This is initially quite surprising, because p itself is playing a role similar to that played by $c \neq 0$ in the original definition of tight closure: in char. p, multiplying by roots of $p = 0$ cannot be helpful! Heitmann has also suggested variant notions in which the multiplier $c^{1/N}$ of small order is only required to multiply u into $(I, p^N)R^+$ (*full extended plus closure*), and a similar notion (*full rank one closure*) where for every rank one $\mathbb{Q}$-valuation v on R^+, for all $\epsilon > 0$ and for all $n > 0$, there exists $d \in R^+ - \{0\}$ with $v(d) < \epsilon$ such that $du \in (I, p^N)R^+$.

Heitmann's result in dimension three yields the direct summand conjecture, and I have been able to use it to prove the existence of big C-M algebras in dim. 3 in a weakly functorial form. In general, if one can construct an algebra that is "almost C-M" in a certain sense, one can actually construct one that is C-M. E.g., suppose that R is a complete local domain, and that B is an extension domain that is a local R-algebra, not necessarily Noetherian, with a $\mathbb{Q}$-valuation that is positive on the maximal ideal of R. Suppose that for every s.o.p. $x_1, \ldots, x_d$ of R, if $u \in B$ and $x_{k+1}u \in (x_1, \ldots, x_k)B$, then there are non-zero elements c_j of arbitarily small order in B such that $c_j u \in (x_1, \ldots, x_k)B$. (In this sense, the x_j are "almost" a regular sequence in B.) Then R has a big C-M algebra.

Heitmann's theorem implies that if R is a complete local domain of mixed char. and of dim. 3, then one may take $B = R^+$, with the elements c_j of arbitrarily small order being roots of p.

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"Closure operations in mixed characteristic"

H. Schoutens has introduced uncountably many (depending on choice of an ultrafilter) tight closure theories in equal char. 0.

∃? big C-M alg. /R ∃? big C-M alg. /S

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \xrightarrow[\text{local}]{} & S \end{array}$$

note: true in equal char.

want: $u \in (u_1, \ldots, u_n)^* \iff$

$$H_m^d \left(\frac{R[X_1, \ldots, X_n]}{u - \sum X_i u_i} \right) \neq 0$$

↳ generic forcing alg.

P. Roberts showed:

$$H^3_{(x_1, x_2, x_3)}\left(\frac{\mathbb{C}[\![x_1, x_2, x_3]\!][Y_1, Y_2, Y_3]}{x_1^2 x_2^2 x_3^2 - \sum_{i=1}^{3} Y_i x_i^3}\right) \neq 0$$

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ Δ_i = minor without i-th column

$\mathbb{C}[\![\Delta_1, \Delta_2, \Delta_3]\!] \longrightarrow \mathbb{C}[\![x_{ij}]\!]$ parasolid

cannot be mapped to a big C-M algebra

note: the above map splits,
but not in char. p, for any p.

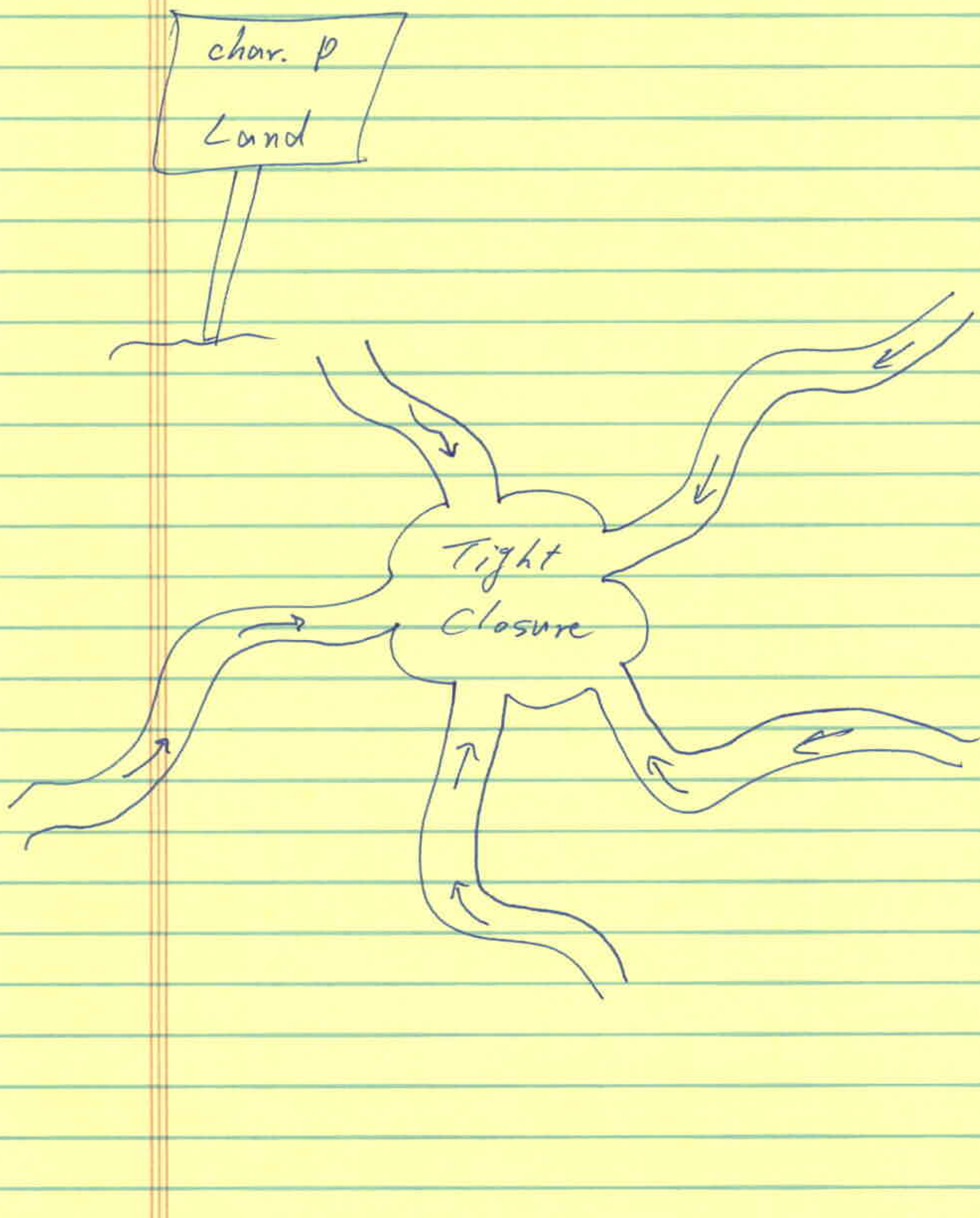
In char. p, R^+ (R local, excellent)

is a big C-M alg.

false in equal char. 0, dim 3

false in mixed char., dim 4

open in mixed char., dim 3


char. p
Land
Tight
Closure