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"F-pure threshold of ideals and its applications to certain singularities"

joint with S. Takagi

$(A, \mathfrak{m})$ Noetherian local, char. $= p > 0$

We always assume A is F-finite.

$F : A \to A$ is finite

$q = p^e$, $[F^e : A \to A] = [A \hookrightarrow A^{1/q}]$

§1 F-pure threshold

Def. $I \subset A$ ideal, $t \in \mathbb{R} > 0$

(1) (A, tI) is F-pure $\iff$

$\forall q \gg 0 \ \exists a \in I^{\lfloor (q-1)t \rfloor}$

$$\begin{array}{ccc} A & \hookrightarrow & A^{1/q} \quad \text{splits} \\ 1 & \longmapsto & a^{1/q} \end{array}$$

(2) (A, tI) is str. F-reg. $\iff$

$\forall c \in A^{\circ} \ \forall q \gg 0 \ \exists a \in I^{\lfloor (q-1)t \rfloor}$

$$A \longrightarrow A^{1/q}$$
$$1 \longmapsto (ca)^{1/q}$$

(3) $\beta_A(I) := \sup\{t : (A, tI) \text{ is F-pure}\}$

the F-pure threshold of I.

Ex. (1)

(A, m) RLR of $\dim A = d$

$(x_1, \ldots, x_d)$ reg. s.o.p. of A

$A \hookrightarrow A^{1/q}$ splits

$1 \longmapsto (x_1 \cdots x_d)^{q-1/q}$

$\beta_A(m) \geq d$ $\quad (\beta_A(m) = d)$

(2) $A = k[[x, y]] \supset I = (f)$

$f = x^a + y^b \quad (a, b \geq 2)$

$\beta_A(I) = \frac{1}{a} + \frac{1}{b}$

$(\because)$ $f^{\lfloor (q-1)(\frac{1}{a} + \frac{1}{b}) \rfloor}$ has $x^i y^j$

$(i \leq q-1, \; j \leq q-1)$

F-rat'l double pts. in dim 2

$A = k[[x, y, z]]/(f)$

(A_n) $f = xy + z^{n+1}$ $\beta_A(m) = 1$

(D_n) $x^2 + y(z^2 + y^{n-2})$ $= \frac{1}{2}$ $(p > 2)$

(E_6) $x^2 + y^3 + z^4$ $= \frac{1}{3}$

(E_7) $x^2 + y(y^2 + z^3)$ $= \frac{1}{4}$

$\Big\}$ $(p > 3)$

(E_8) $x^2 + y^3 + z^5$ $= \frac{1}{6}$ $(p > 5)$

non-rat'l double pts.

$f = x^2 + y^3 + z^6$ $\beta_A(m) = 0$

$x^2 + y^3 + z^7$ $= -\frac{1}{6}$

Rmk.

① If $A \hookrightarrow A^{1/q}$ splits $\Longrightarrow$
$1 \mapsto a^{1/q}$

$A \hookrightarrow A^{1/q}$ splits i.e. A is F-pure.
$1 \mapsto 1$

② $I \supset I' \Longrightarrow \beta_A(I) \geq \beta_A(I')$

③ $\beta_A(I) = \beta_A(\bar{I})$

note $(A, t\bar{I})$ is F-pure $\not\Longrightarrow$

(A, tI) is F-pure.

④ $\beta_A(I^r) = \frac{1}{r} \beta_A(I)$

Ex. $A = k[[x_1, \dots, x_d]] \supset B = A^{(r)} = k[[r\text{-forms}]]$

$(A, m) \hookleftarrow (B, n)$

$\beta_B(n) = \beta_A(nA) = \beta_A(m^r) = \frac{d}{r}$

How to compute?

<u>Lemma</u> (Hochster – Roberts)

$$A \hookrightarrow A^{1/q} \quad \text{splits} \qquad \Longleftrightarrow$$
$$1 \longmapsto a^{1/q}$$

$$A \otimes E \longrightarrow A^{1/q} \otimes E \qquad \text{inj.}$$

$$1 \otimes x \longmapsto a^{1/q} \otimes x$$

In Gor. case, J: param. ideal,

$A/J \ni z$: socle

find a s/t $a \cdot z^q \notin J^{[q]}$

<u>Prop.</u> (A, m) local

Assume A is normal, $\mathbb{Q}$ Gor.

$f: X \longrightarrow \operatorname{Spec}(A)$ proj. birat'l

X normal

$I\mathcal{O}_X = \mathcal{O}_X(-Z)$ $\quad Z = \sum_i n_i E_i$

$E = \bigcup E_i$ exc. divisor

$$K_X = f^* K_A + \sum \underset{\mathbb{Q}}{\underset{\cap}{k_i}} E_i$$

If (A, tI) is F-pure than

$k_i \geq -1 + t n_i \quad (\forall i)$ i.e.

(A, tI) is F-pure

then (A, tI) is log-can.

In char $= 0$ and X: res'l of A

A is terminal $\iff$ $\forall\ k_i > 0$

log term. $\iff$ > -1

log-can. $\iff$ ≥ -1

(A, tI) is log-can. $\iff$ $k_i - t n_i \geq -1$

$lc(I) = \sup\{t : (A, tI) \text{ is log-can.}\}$

Prop. If (A, m) : ess. of fin. type $/k$, $\mathrm{char}(k) = 0$; if (A, tI) is log-term. then for every $p \gg 0$ the reduction of (A, tI) to char $= p$ is F-reg.

$$\boxed{lc(I) = \beta_A(I)}$$

§2 big values of $\beta_A(I)$

Thm. (A, m) : dim $= d$, $I \subsetneq A$ ideal

$J \subset I$: min red. of I

(1) $\forall I$, $\beta_A(I) \le ht(I)$

if $\beta_A(m) > d-1$ then A is reg.

(1') if $\beta_A(I) > d - r$ $(r \in \mathbb{Z}_{>0})$

$\Longrightarrow I^r \subset J$.

(2) if A is Gor. with $\beta_A(m) = d-1$

$\Longleftrightarrow \exists\ (x_1, \dots, x_{d-1}) \subset m$ s/t

$A/(x_1, \dots, x_{d-1}) \cong k[\![x, y]\!]/(xy)$ $\rightarrow \beta_A(m) = 0.$

(A is compound (A_n)-sing.)

(3) if A is $\mathbb{Q}$-Gor. $\left[(K_A)^{(r)} = \omega A\right]$

$(r > 1)$ then $\beta_A(m) \le d - 1 - \frac{1}{r}$

Pf. $(1') \Longrightarrow (1)$

Briançon-Skoda Thm.

Assume $I^n = I^{n-q} J^q$ $\forall (n \gg 0)$

Take $u \in I^{(d-r)q+s}$ $\qquad A \hookrightarrow A^{1/q}$ splits, $A^{1/q} \to A$, $1 \mapsto u^{1/q}$

Take $x \in I^r$ suffices to show

$u x^q \in J^{[q]}$, $\quad u x^q \in J^{dq} \subset J^{[q]}$ //

~~Pnk~~ (2)

use Lemma: $I \cong K_A$ divisorial,

Assume $A/I =: B$ F-pure

$I^{(r)} = wA \quad (r \geq 1)$

$\operatorname{ord}(I) := \frac{1}{r} \operatorname{ord}_m (w)$

$\Longrightarrow \beta_A(m) \geq \beta_B(m/I) + \operatorname{ord}(I)$

Pf. of (2) if $\beta_A(m) = d-1$, A is a double pt.

If $A = k[[x_1, \ldots, x_{d+1}]]/(f)$

$f = x_1^2 + \underbrace{g(x_2, \ldots, x_{d+1})}_{\in m^3} \Longrightarrow \beta_A(m) \leq d - \frac{3}{2}$.

<u>Rmk</u> what if A is NOT $\mathbb{Q}$-Gor.?

$A = k[[x_{ij} : \begin{smallmatrix} i = 1, \ldots, d-1 \\ j = 1, 2, \ldots \end{smallmatrix}]] / I_2(x_{ij})$

$\beta_A(m) = d-1$

Question: If $\beta_A(m) = d-1$, then $A \cong$ this ex?

§3 negative $\beta_A(I)$

char p, char. of sing. in char $= 0$.

(Hara – Smith, Mehta – Srinivas)

A is a rat'l sing. $\Longleftrightarrow$ A is of F-rat'l type

A is log-term $\Longleftrightarrow$ A is of str. F-reg type

and $\mathbb{Q}$-Gor.

A is log-can. $\Longleftarrow$ A is of F-pure type

$\underset{\text{difficult(?)}}{\Longrightarrow}$ and $\mathbb{Q}$-Gor.

<u>Def</u> $\beta_A(I) \geq -n+r \quad (n \in \mathbb{Z}_{>0},\ r \in \mathbb{Q}_{>0})$

$\Longleftrightarrow \quad \forall\ b \in I^n \quad \exists\ q \gg 0$

$\exists\ a \in I^{\lfloor r(q-1) \rfloor}$ s/t

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "b"'] & & A^{1/q} \arrow[dl, "\exists \varphi"] \\
& A &
\end{tikzcd}
\qquad 1 \longmapsto a^{1/q}$$

i.e. $\varphi(a^{1/q}) = b$.

How to calculate?

<u>Lemma</u> Diagram as above

$\exists f$: splitting $\iff [0:b]_E \supset \ker[a^{1/q} : E \to E \otimes A^{1/q}]$

with $E = E_A(A/m)$.

<u>Def.</u> A is qs-F-pure $\iff \beta_A(m) \geq 0$.

~~If $\beta_A(I) \geq 0$~~

We can make the same conclusion about discrepancy of divisors.

<u>Conj</u> A is log-can. $\iff$ A is of qs-F-pure type and $\mathbb{Q}$-Gor.

<u>Ex.</u> $A = \bigoplus_{n \geq 0} A_n$ graded, $A_0 = k$ field.

① if A is Gor. $a(A) \leq 0$ and if

$\tau_A = m$ (test ideal) $\Rightarrow$

A is qs-F-pure.

② if $a(A) > 0$ then A is <u>not</u> qs-F-pure.