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"Computing instanton numbers of plane curve singularities"

1. Invariants of plane curves

2. joint with Gasparim, Instanton numbers

set-up: $R = \mathbb{C}[X,Y]$, $f \in R$, $P = (0,0)$

1. Multiplicity: $f = \sum f_{il} X^i Y^l$

$$d = \min\{i+l : f_{il} \neq 0\}$$

In case f is homog., graded betti numbers

of $R/(f)$: $0 \to R(-d) \to R \to R/(f) \to 0$

Samuel polynomial

$$\lambda\left(R/(X,Y)^n + (f)\right) \overset{n \gg 0}{=} dn + \left(1 - \binom{d-1}{2}\right)$$

arithmetic genus

2. Milnor number of f at P

$$\lambda\left(\left(R\Big/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right)_{(x,y)}\right)$$

example: $f = x^5 - x^2y^2 + y^5$

$$\frac{\partial f}{\partial x} = 5x^4 - 2xy^2$$

$$\frac{\partial f}{\partial y} = 5y^4 - 2x^2y$$

$$\lambda\left(R\Big/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right)$$

first globally:

$$xy^3(-4 + 25xy)$$

locally at (x,y): $-4 + 25xy \Rightarrow \lambda = 11$

3. Tjurina number

$$\lambda\left(\left(R\Big/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\right)\right)_{(x,y)}\right)$$

$\text{Tjurina}(x^5 - x^2y^2 + y^5) = 10$

4. Delta invariant

f: reduced

$A = R/(f)$ reduced

$\overline{A}$ = normalization, module-finite over A

$(\bar{A}/A)_{(x,y)}$ finite length

$$\delta_p(f) = \lambda_A\left((\bar{A}/A)_{(x,y)}\right)$$

Fact: $\delta_p(f) = \sum \frac{r_Q(r_Q-1)}{2}$, summing

over all points Q infinitely near to P

and r_Q multiplicity of the strict transform

of f at Q.

(Q is inf. near P if Q lives in some

blow-up and maps down to P.)

example $f = x^5y - y^4$, $P=(0,0)$, $r_p = 4$

blow-up at P

On $\mathbb{C}[\frac{x}{y}, y]$, strict transform:

$(\frac{x}{y})^5 y^2 - 1$ nonsingular

On $\mathbb{C}[x, \frac{y}{x}]$, strict transform

$$x^2 \frac{y}{x} - \left(\frac{y}{x}\right)^4$$

$$Q \longrightarrow r_Q = 3$$

$$\delta = \frac{1}{2}(4\cdot 3 + 3\cdot 2) = 9$$

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2. Instantons

algebraic side: Hitchin - Kobayashi correspondence

$$\{\text{instantons}\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{holomorphic} \\ \text{bundles} \\ \text{with } 1^{st} \text{ Chern class } 0\end{array}\right.$$

topological charge

$$\left\{\begin{array}{l}\text{rank 2} \\ \text{instantons} \\ \text{on } \tilde{\mathbb{C}}^2\end{array}\right\} \overset{(\text{King})}{\longleftrightarrow} \left\{\begin{array}{l}\text{holomorphic rank 2} \\ \text{bundles on } \mathbb{C}^2 \\ \text{with first Chern class } 0\end{array}\right\}$$

$$\underset{\substack{\text{blow-up} \\ \text{at } (0,0)}}{\tilde{\mathbb{C}}_2} \xrightarrow{\pi} \mathbb{C}_2$$

(on compact spaces top. charges correspond to 2^{nd} Chern class) charge $\tilde{E} = c_2(\tilde{E}) - c_2\left((\pi_* \tilde{E})^{\vee\vee}\right)$

$$= \underbrace{\lambda(Q)}_{\text{width}} + \underbrace{\lambda(R^1 \pi_* \tilde{E})}_{\text{height}}$$

$$0 \to \pi_* \tilde{E} \to (\pi_* \tilde{E})^{\vee\vee} \longrightarrow Q \to 0$$

$$\tilde{E}|_{\ell} = \mathcal{O}(j) \oplus \mathcal{O}(-j), \quad j \geq 0$$

$j =$ splitting type of $\tilde{E}$

$\tilde{E}$ = extension of $\mathcal{O}(-j)$ and $\mathcal{O}(j)$

$p \in \mathrm{Ext}^1_{\tilde{\mathbb{C}}^2}(\mathcal{O}(-j), \mathcal{O}(j))$

polynomial

for j and p, enough to compute width and height

$\tilde{\mathbb{C}}^2$ charts $U \cup V$, $U = \{(z, u)\}$, $V = \{(z^{-1}, zu)\}$

$\downarrow$

$\mathbb{C}^2$

$x \longmapsto u$

$y \longmapsto uz$

under a change of basis

$$\tilde{E} \xrightarrow{\text{transition matrix}} \begin{bmatrix} z^j & p(z, z^{-1}, u) \\ 0 & z^{-j} \end{bmatrix}$$

computation

$$\widehat{(\pi_* \tilde{E})}_0 = \varprojlim H^0(\ell_n | \tilde{E}|_{\ell_n})$$

$\hookrightarrow \ell_n$: n^{th} infinitesimal nbhd.

looking for

$$\sum_{i,\ell \geq 0} \begin{pmatrix} a_{i\ell} \\ b_{i\ell} \end{pmatrix} u^i z^\ell \quad \text{with} \quad \begin{bmatrix} z^j & p \\ 0 & z^{-j} \end{bmatrix} \sum \begin{pmatrix} a_{i\ell} \\ b_{i\ell} \end{pmatrix} u^i z^\ell$$

$\hookrightarrow$ column vector

is hol. in V

second coord.

$$\sum b_{i\ell} (uz)^i z^{\ell - j - i}$$

need $\ell - j - i < 0$

$b_{i\ell} = 0$ if $\ell > i + j$

but:

$$\varprojlim H^0(\ell_n | \tilde{E}|_{\ell_n}) = H^0(\ell_{2j-2}, \tilde{E}|_{\ell_{2j-2}})$$

Obtain $\pi_* E$ as a module over $R = \mathbb{C}[x,y]$

$= \text{coker}(A)$, $\quad R^n \xrightarrow{A} R^m \to M \to 0$

With $M = \pi_* E$, $\quad M^\vee =$ kernel transpose A

$$Q = \frac{M^{\vee\vee}}{M} = \frac{\text{kernel transpose presentation kernel transpose } A}{\text{image transpose gens kernel transpose } A}$$