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"Computing instanton numbers of plane curve singularities"

1. Invariants of plane curves

2. joint with Gusparim, Instanton numbers

set-up: $R = \mathbb{C}[X, Y]$, $f \in R$, $P = (0, 0)$

1. Multiplicity: $f = \sum f_{i,l} x^i y^l$

$$d = \min \{i+l : f_{i,l} \neq 0\}$$

In case f is homog., graded betti numbers

$$\text{of } R/(f) : 0 \rightarrow R(-d) \rightarrow R \rightarrow R/(f) \rightarrow 0$$

Samuel polynomial

$$\chi \left(\frac{R}{\mathbb{C}[X, Y]^n + (f)} \right) \stackrel{n \gg 0}{=} dn + \left(1 - \binom{d-1}{2} \right)$$

with the trc genus

2. Milnor number of f at P

$$\lambda \left(\left(\frac{R}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)} \right)_{(x,y)} \right) \quad \text{example: } f = x^5 - x^2y^2 + y^5$$

$$\frac{\partial f}{\partial x} = 5x^4 - 2xy^2$$

$$\frac{\partial f}{\partial y} = 5y^4 - 2x^2y$$

$$\lambda \left(\frac{R}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)} \right) \quad \text{first globally:}$$

$$xy^3 (-4 + 25xy)$$

$$\text{locally at } (x,y) : -4 + 25xy \Rightarrow \lambda = 11$$

3. Tjurina number

$$\lambda \left(\left(\frac{R}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right)} \right)_{(x,y)} \right)$$

$$\text{Tjurina } (x^5 - x^2y^2 + y^5) = 10$$

4. delta invariant

f : reduced

$$A = R/(f) \quad \text{reduced}$$

\bar{A} = normalization, module-finite over A

$(\bar{A}/A)_{(x,y)}$ finite length

$$S_p(f) = \sum_Q \left((\bar{A}/A)_{(x,y)} \right)$$

Fact: $S_p(f) = \sum \frac{r_Q(r_Q-1)}{2}$, summing

over all points Q infinitely near to P
and r_Q multiplicity of the strict transform
of f at Q .

(Q is inf near P if Q lives on some
blow-up and maps down to P .)

example $f = x^5y - y^4$, $P=(0,0)$, $r_P=4$

blow-up at P

On $\mathbb{C}[\frac{x}{y}, y]$, strict transform:

$$\left(\frac{x}{y}\right)^5 y^2 - 1 \quad \text{non-singular}$$

On $\mathbb{C}[x, \frac{y}{x}]$, strict transform

-4-

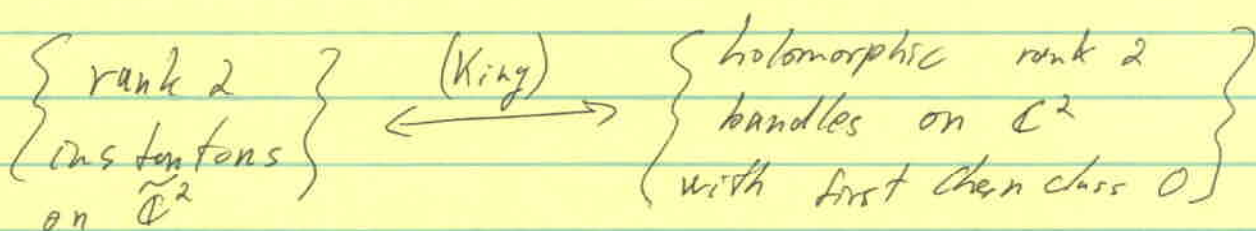
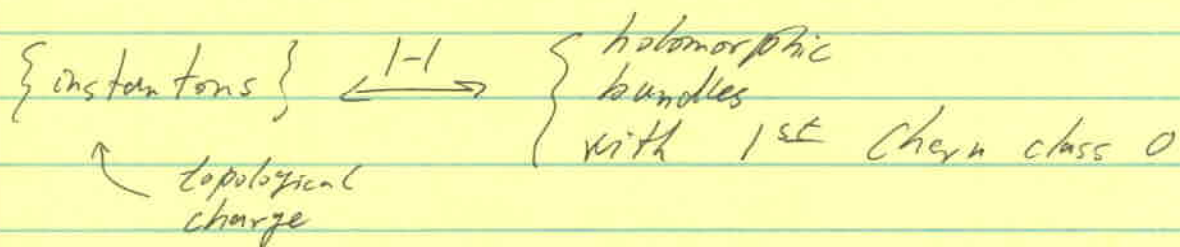
$$x^2 \frac{4}{x} - \left(\frac{4}{x}\right)^4$$

$$Q \rightarrow r_Q = 3$$

$$\delta = \frac{1}{2} (4 \cdot 3 + 3 \cdot 2) = 9$$

2. Instantons

algebraic side: Hitchin - Kobayashi correspondence



$$\tilde{\mathbb{C}}^2 \xrightarrow{\pi} \mathbb{C}^2$$

blow-up
at (0,0)

(on compact spaces top. charges correspond to 2nd Chern class)

change $\tilde{E} = c_2(\tilde{E}) - c_2(\pi_* \tilde{E}^{\vee\vee})$

width $= \lambda(Q) + \lambda(R' \pi_* \tilde{E})$ height

$$0 \rightarrow \pi_* \tilde{E} \rightarrow (\pi_* \tilde{E})^{\vee\vee} \rightarrow Q \rightarrow 0$$

$$\tilde{E}|_x = \mathcal{O}(j) \oplus \mathcal{O}(-j), \quad j \geq 0$$

$j =$ splitting type of \tilde{E}

$\tilde{E} =$ extension of $\mathcal{O}(-j)$ and $\mathcal{O}(j)$

$$p \in \text{Ext}_{\mathbb{C}_2}^1(\mathcal{O}(-j), \mathcal{O}(j))$$

polynomial

for j and p , enough to compute width and height

$\tilde{\mathbb{C}}_2$ charts $U \cup V$, $U = \{(z, u)\}$
 $V = \{(z^{-1}, zu)\}$



$$\mathbb{C}_2$$

under a change of basis

$$\tilde{E} \xrightarrow{\text{transition matrix}} \begin{bmatrix} z^j & p(z, z^{-1}, u) \\ 0 & z^{-j} \end{bmatrix}$$

$$x \mapsto u$$

$$y \mapsto uz$$

computation

$$\widehat{(\pi_* \tilde{E})}_0 = \varprojlim H^0(L_n | \tilde{E}|_{L_n})$$

↳ L_n : n^{th} infinitesimal nbhd.

looking for

$$\sum_{i,l \geq 0} \begin{pmatrix} a_{il} \\ b_{il} \end{pmatrix} u_i z^l \quad \text{with} \quad \begin{pmatrix} z^j & p \\ 0 & z^{-l} \end{pmatrix} \sum \begin{pmatrix} u_{il} \\ b_{il} \end{pmatrix} u_i z^l$$

↳ column vector is hol. in V

second coord.

$$\sum b_{il} (u_i z)^i z^{l-j-i}$$

need $l-j-i < 0$
 $b_{il} = 0$ if $l > i+j$

but:

$$\varprojlim H^0(L_n | \tilde{E}|_{L_n}) = H^0(L_{2j-2}, \tilde{E}|_{L_{2j-2}})$$

Obtain $\pi_* E$ as a module over $R = \mathbb{C}[X, Y]$

$$= \text{coker}(A), \quad R^n \xrightarrow{A} R^m \rightarrow M \rightarrow 0$$

With $M = \pi_* E$, $M^\vee = \text{kernel transpose } A$

$$Q = \frac{M^{\vee\vee}}{M} = \frac{\text{kernel transpose presentation kernel transpose } A}{\text{image transpose } A \text{ kernel transpose } A}$$