WILLIAM DWYER (p. 1) 2-3 11:00 AM

Greenlees / Iyengar

Notation "Ring R" ordinary ring $\subseteq$ DG Algebra ($\subseteq$ Ring Spectra) $\mathbb{Z}$ S^0

"Module M" ordinary module $\subseteq$ DG Module ($\subseteq$ Module Spectra).

M, N R-modules $\mathrm{Hom}_R(M,N)$ = derived chain complex of maps between M and N

$M \otimes_R N$ = derived tensor product of M and N

If R an ordinary ring, M, N ord. modules,

$$H_i(\mathrm{Hom}_R(M,N)) = \mathcal{E}xt_R^{-i}(M,N)$$

$$H_i(M \otimes_R N) = \mathrm{Tor}_i^R(M,N)$$

R ordinary ring, M an R-module

$\mathrm{End}_R(M)$ = derived endomorphism complex of $M = \mathrm{Hom}_R(M,M)$

$H_i(\mathrm{End}_R(M)) \simeq \mathcal{E}xt_R^{-i}(M,M)$ $\mathrm{End}_R(M)$ is a DGA

$R \to k$ map of rings

$\mathcal{E} = \mathrm{End}_R(k)$

Focus on two constructions for an R-module M.

$E(M) = \mathrm{Hom}_R(k, M)$ <u>right</u> $\mathcal{E}$-module

$\Gamma(M) = \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \xrightarrow{\text{eval}} M$

$\qquad \| $

$\qquad E(M) \otimes_{\mathcal{E}} k$

<u>Remark</u> Under reasonable hypotheses (*)

$E(\Gamma(M)) \approx E(M)$

$\Gamma(\Gamma(M)) \approx \Gamma(M)$

(WD p.2)

$$\mathrm{Hom}_R(k,\Gamma(M)) \simeq \mathrm{Hom}_R(k,\mathcal{E}(M)\otimes_{\mathcal{E}} k) \overset{?}{=} \mathcal{E}(M)\otimes_{\mathcal{E}} \overbrace{\mathrm{Hom}_R(k,k)}^{\mathcal{E}} = \mathcal{E}(M)$$

<u>Remark</u> (Standard alg. context) R - local noeth. comm. ring, k = residue class field ($R\to k$ reduction mod I)
<u>Then</u> $\Gamma(M) \approx H_I^0(M)$. $H_i\Gamma(M) \simeq H_I^{-i}(M)$
(Note $(*)$ is satisfied.)

<u>Ex</u> Get spec. sequence

$$\mathrm{Tor}_p^{\mathrm{Ext}_R^*(k,k)}(\mathrm{Ext}_R^*(k,M),k)_q \Rightarrow H_I^{-(p+q)}(M)$$

② Pondriagin dualizer
Given $R\to k$, a "Pondriagin dualizer" is an R module J s.t.
$\mathrm{Hom}_R(k,J) \approx \mathrm{Hom}_k(k,k) = k$
$\mathcal{E}(J)$
Say that $J\sim J'$ if they are related by the equivalence relation generated by having a map $J\to J'$ inducing $\mathrm{Hom}_R(k,J)\simeq \mathrm{Hom}_R(k,J')$

Recall $\mathcal{E} = \mathrm{End}_R(k)$

$$\left(\begin{array}{c}\text{Equiv. classes}\\ \text{of Pond. dualizers}\end{array}\right) \xrightarrow{\mathrm{Hom}_R(k,-)} \left(\begin{array}{c}\text{Right}\\ \mathcal{E}\text{-module}\\ \text{on } k\end{array}\right)$$

Under some reasonable hypotheses, this map is bijective
<u>Ex</u> Standard alg. context $R\to k$
- $H_i(\mathcal{E}) = \mathrm{Ext}_R^{-i}(k,k)$
- There is only one way to pick a right $\mathcal{E}$-structure on k
- only one equiv. class of Pond. dualizers, represented by $k\otimes_{\mathcal{E}} k$
- Pond dual. is given by $I(k)$ (inj. hull), $I(k)\simeq k\otimes_{\mathcal{E}} k$

(WD p 3)

$R = \mathbb{Z}$, $k = \mathbb{Z}/p$ $\quad H_* \mathcal{E} = \begin{cases} \mathbb{Z}/p & * = 0 \\ \mathbb{Z}/p & * = -1 \end{cases}$ $\quad \mathbb{Z}/p^\infty \simeq \mathbb{Z}/p \otimes_{\mathcal{E}} \mathbb{Z}/p$

Ex $k = \mathbb{Z}/p$, $R = k[G]$ G a finite group

$H_i \mathcal{E} = H_i(\mathcal{E}nd_R(k,k)) \simeq \mathcal{E}xt_R^{-i}(k,k) \simeq H^{-i}(BG;k)$

$\mathcal{E} \simeq C^*(BG;k)$

Again, only one right action of $\mathcal{E}$ on k, only one class of Pond. dual.

$J \simeq k \otimes_{\mathcal{E}} k \simeq C^*(\Omega(BG)^\wedge_p, \mathbb{Z}/p)$

Kunneth s.s. $\mathrm{Tor}_*^{H^*BG}(\mathbb{Z}/p, \mathbb{Z}/p) \longrightarrow H_*(J)$ is an Eilenberg-Moore spec. seq^{n}.

Ex X 1-connected finite complex, $k = \mathbb{Z}/p$, $R = C^*(X;k)$

$\mathcal{E} = \mathcal{E}nd_R(k) \simeq C_*(\Omega X; k)$

Again, only one right action of $\mathcal{E}$ on k, only one Pond. dual.

$J \approx k \otimes_{\mathcal{E}} k \simeq C_*(X,k)$ (Actually, k-dual of R)

③ $R \to k$ is Gorenstein if (up to suspension) R itself is a Pond. dualizer.

$\mathrm{Hom}_R(k,R) \simeq \Sigma^d k \qquad \mathrm{Hom}_R(k, \Sigma^{-d} R) \simeq k$

If $R \to k$ is Gor, then $\left(\begin{smallmatrix}\text{Pond. dual}\\ \text{of } R \to k\end{smallmatrix}\right) \xleftrightarrow{\text{agree up to a suspension}} \Gamma(R)$

Ex Standard alg. context $R \to k$

$R \to k$ is Gor. if $\mathcal{E}xt_R^i(k,R) = \begin{cases} k & i = d \\ 0 & \text{otherwise} \end{cases}$

i.e., $R \to k$ is Gorenstein in the usual sense

$\Sigma^{-d} I(k) \simeq \Gamma(R) \simeq H_I^0(R)$

Ex $k = \mathbb{Z}/p$, $R = k[G]$, G a finite group

$H_i(\mathrm{Hom}_R(k,R)) \simeq \mathcal{E}xt_R^{-i}(k,R) \simeq \begin{cases} k & i = 0 \\ 0 & \text{otherwise} \end{cases}$ $\Bigg\}$ Put off discussing consequences

(WD p.4)

<u>Ex</u> $k = \mathbb{Z}/p$, $R = C^*(X, k)$, X 1-connected finite

$\mathrm{Hom}_R(k, R) = C_*$(some specific geometric object associated to X)

$\simeq \Sigma^{-d} k \longleftrightarrow X$ is a Poincaré duality space of formal dim d

$R \to k$ is GOR $\longleftrightarrow X$ satisfies P.D.

Pond dualizer $= C_* X$ $\xleftrightarrow[\text{suspension}]{\text{agree up to}}$ $\Gamma(R)$ $R = C^* X$ Poincaré duality statement

④ <u>Odd fact</u> Under reasonable hypotheses

$R \to k$ is Gorenstein $\longleftrightarrow \mathcal{E} \to k$ is

Back to Group Ring Case

$k[G] \to k$ is GOR

$C^*(BG; k) = \mathcal{E} \to k$ is GOR

Pond. dual. for $\mathcal{E} \to k$ $\simeq$ $\Gamma(\mathcal{E})$

$\Gamma \longleftrightarrow$ local cohomology

$C_*(BG)$ ~~[illegible]~~

I = any ideal in H^*BG

$H_*(BG) \longleftarrow H^*_I(H^*(BG; k))$

$R \to k$ $\mathrm{Hom}_R(k, R) \simeq \Sigma^d k$