

Greenlees/Syngar
 Notation "Ring R "
 "Module M "

ordinary ring \in DG Algebra $\xrightarrow{\quad} \mathbb{Z} \quad S^0$
 ordinary module \in DG Module $\left(\begin{array}{l} \in \text{Ring Spectra} \\ \in \text{Module Spectra} \end{array} \right)$

M, N R -modules $\text{Hom}_R(M, N) =$ derived chain complex of maps
 between M and N

$M \otimes_R N =$ derived tensor product of M and N

If R an ordinary ring, M, N ord. modules,
 $H_i(\text{Hom}_R(M, N)) = \text{Ext}_R^{-i}(M, N)$

$$H_i(M \otimes_R N) = \text{Tor}_i^R(M, N)$$

R ordinary ring, M an R -module

$\text{End}_R(M) =$ derived endomorphism complex of $M = \text{Hom}_R(M, M)$

$$H_i(\text{End}_R(M)) \cong \text{Ext}_R^{-i}(M, M) \quad \text{End}_R(M) \text{ is a DGA}$$

$R \rightarrow k$ map of rings

$$E = \text{End}_R(k)$$

Focus on two constructions for an R -module M .

$$E(M) = \text{Hom}_R(k, M) \text{ right } E\text{-module}$$

$$\Gamma(M) = \text{Hom}_R(k, M) \otimes_E k \xrightarrow{\text{eval}} M$$

$E(M) \otimes_E k$

Remark Under reasonable hypotheses (*)

$$E(\Gamma(M)) \cong E(M)$$

$$\Gamma(\Gamma(M)) \cong \Gamma(M)$$

(WD p. 2)

$$\mathrm{Hom}_R(k, \Gamma(M)) \cong \mathrm{Hom}_R(k, E(M) \otimes_E k) \stackrel{?}{=} E(M) \otimes_E \overbrace{\mathrm{Hom}_R(k, k)}^E = E(M)$$

Remark (Standard alg. context) R -local noeth. comm. ring,
 k = residue class field ($R \rightarrow k$ reduction mod \mathfrak{I})

Then $\Gamma(M) \cong H_{\mathfrak{I}}^0(M)$. $H_i \Gamma(M) \cong H_{\mathfrak{I}}^{-i}(M)$

(Note (*) is satisfied.)

Ex Get spec. sequence

$$\mathrm{Tor}_p^{\mathrm{Ext}_R^+(k, k)} (\mathrm{Ext}_R^+(k, M), k)_q \Rightarrow H_{\mathfrak{I}}^{-(p+q)}(M)$$

② Poincaré dualizer

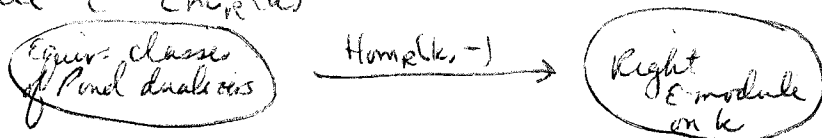
Given $R \rightarrow k$, a "Poincaré dualizer" is an R -module J s.t.

$$\mathrm{Hom}_R(k, J) \cong \mathrm{Hom}_k(k, k) = k$$

$E(J)$

Say that $J \sim J'$ if they are related by the equivalence relation generated by having a map $J \rightarrow J'$ inducing $\mathrm{Hom}_R(k, J) \cong \mathrm{Hom}_R(k, J')$

Recall $E = \mathrm{End}_R(k)$



Under some reasonable hypotheses, this map is bijective

Ex Standard alg. context $R \rightarrow k$

$$- H_i(E) = \mathrm{Ext}_R^i(k, k)$$

- There is only one way to pick a right E -structure on k

- only one equiv. class of Poinc. dualizers, represented by $k \otimes_E k$

- Poinc. dual. is given by $\mathbb{I}(k)$ (inj. hull), $\mathbb{I}(k) \cong k \otimes_E k$

(WD p3)

$$R = \mathbb{Z}, k = \mathbb{Z}/p \quad H_* E = \begin{cases} \mathbb{Z}/p & * = 0 \\ \mathbb{Z}/p & * = -1 \end{cases} \quad \mathbb{Z}/p \simeq \mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/p$$

Ex $k = \mathbb{Z}/p, R = k[G]$ G a finite group
 $H_i E = H_i(\text{End}_R(k, k)) \simeq \text{Ext}_R^{-i}(k, k) \simeq H^{-i}(BG; k)$
 $E \simeq C^*(BG; k)$

Again, only one right action of E on k , only one class of Pond. dual.

$$J \simeq k \otimes_{\mathbb{Z}} k \simeq C^*(\hat{\Omega}(BG)_p, \mathbb{Z}/p)$$

Kunnetth s.s. $\text{Tor}_*^{H^*BG}(\mathbb{Z}/p, \mathbb{Z}/p) \rightarrow H_*(J)$ is an Eilenberg-Moore spec. seqⁿ.

Ex X 1-connected finite complex, $k = \mathbb{Z}/p, R = C^*(X; k)$

$$E = \text{End}_R(k) \simeq C_*(\Omega X; k)$$

Again, only one right action of E on k , only one Pond. dual.

$$J \simeq k \otimes_{\mathbb{Z}} k \simeq C_*(X, k) \quad (\text{Actually, } k\text{-dual of } R)$$

③ $R \rightarrow k$ is Gorenstein if (up to suspension) R itself is a Pond. dualizer.

$$\text{Hom}_R(k, R) \simeq \Sigma^d k$$

$$\text{Hom}_R(k, \Sigma^{-d} R) \simeq k$$

If $R \rightarrow k$ is Gr, then (Pond. dual. of $R \rightarrow k$) $\xleftarrow{\text{agree up to a suspension}}$ $\Gamma(R)$

Ex Standard alg. context $R \rightarrow k$

$$R \rightarrow k \text{ is Gr. if } \text{Ext}_R^i(k, R) = \begin{cases} k & i = d \\ 0 & \text{otherwise} \end{cases}$$

i.e., $R \rightarrow k$ is Gorenstein in the usual sense

$$\Sigma^{-d} \text{II}(k) \simeq \Gamma(R) \simeq H_{\Gamma}^0(R)$$

Ex $k = \mathbb{Z}/p, R = k[G]$ G a finite group

$$H_i(\text{Hom}_R(k, R)) \simeq \text{Ext}_R^{-i}(k, R) \simeq \begin{cases} k & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

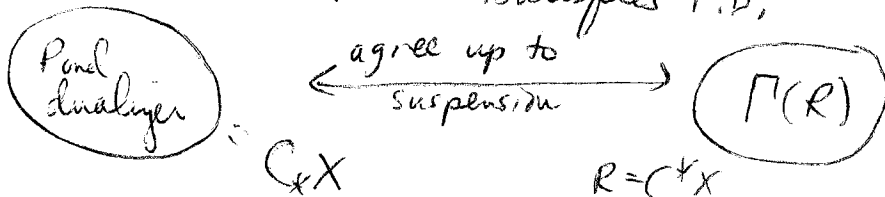
} Put off discussing consequences

(WD p.4)

Ex $k = \mathbb{Z}/p$, $R = C^*(X, k)$, X 1-connected finite
 $\text{Hom}_R(k, R) = C_*(\text{some specific geometric object associated to } X)$

$\cong \sum^{-d} k \iff X$ is a Poincaré duality space of formal dim d

$R \rightarrow k$ is GOR $\iff X$ satisfies P.D.



Poincaré duality statement

④ Odd fact Under reasonable hypotheses
 $R \rightarrow k$ is Gorenstein $\iff E \rightarrow k$ is

Back to Group Ring Case

$k[G] \rightarrow k$ is GOR

$H^*(BG; k) = E \rightarrow k$ is GOR

Poncaré dual. for $E \rightarrow k$

$\cong \Gamma(E)$

$\Gamma \iff$ local cohomology

$C_*(BG)$

$\mathfrak{I} =$ any ideal in H^*BG

$H_*(BG) \longleftarrow H_{\mathfrak{I}}^*(H^*(BG; k))$

$R \rightarrow k$ $\text{Hom}_R(k, R) \cong \sum^d k$