REPRESENTATION THEORY OF CM RNGS

MSRI, February 3, 2003

§0. Introduction.

• Throughout, $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring of dimension d. Modules are always assumed to be finitely generated. ${}_RM$ is MCM provided $\operatorname{depth}(M) = d$.

We are interested in direct-sum decompositions of MCM modules.

Existence. When are there only finitely many indecomposable MCM R-modules — *finite representation type* (**FRT**)? More generally, when is there a bound on the "size" (multiplicity) of the indecomposable modules — *bounded representation type* (**BRT**)?

Uniqueness. Until recently, these questions have been considered only for complete (or at least Henselian) local rings, where one has the *Krull-Schmidt Theorem* (uniqueness of representation as a direct sum of indecomposable MCM modules). Here we will discuss primarily the general situation, where Krull-Schmidt can fail, and we will describe exactly how badly it can fail.

OUTLINE.

§1. The complete equicharacteristic case.
§2. FRT for R vs. $\hat{R}$.
§3. The positive normal monoid add(M)
§4. The number of distinct indecomposable summands.
§5. BRT and Brauer-Thrall.

§1. Complete equicharacteristic rings with FRT.

Theorem. (Buchweitz, Greuel, Schreyer, Knörrer, 1987) *Let* $R = \mathbb{C}[[x_0, \ldots, x_d]]/(f)$, *where* $\mathbb{C}$ *is an algebraically closed field of characteristic* $\neq 2, 3, 5$. *Then* R *has FRT if and only if* $R \cong \mathbb{C}[[x, y, x_2, \ldots, x_d]]/(g + x_2^2 + \cdots + x_d^2)$, *where* $g \in \mathbb{C}[x, y]$ *is one of*

(A_n) $x^{n+1} + y^2 \quad (n \geq 1)$
(D_n) $x(x^{n-2} + y^2) \quad (n \geq 4)$
(E_6) $x^4 + y^3$
(E_7) $y(x^3 + y^2)$
(E_8) $x^5 + y^3$

The classification has been worked out also in characteristics $2, 3, 5$, but the normal forms become much more complicated, particularly in characteristic 2. These results are particularly satisfying in view of the following fact:

Theorem. (Herzog, 1978) *Assume* R *(local, CM) has FRT. Then* $\hat{R}$ *is a hypersurface.*

What about non-Gorenstein rings with FRT? In dimension two, we have the rings $\mathbb{C}[[X, Y]]^G$, where G is a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$. There are two three-dimensional examples, due to Auslander and Reiten (1989). There appear to be no known examples of dimension four or higher.

§2. Ascent and descent of FRT.

Theorem. (Leuschke and ______, 2000; conjectured by Schreyer in 1985)

(1) *If* $\hat{R}$ *has FRT, so has* R. *The converse holds if* R

is excellent.

(2) *Let k be a perfect field with algebraic closure K, and let I be an ideal in the formal power series ring $k[[x_1, \ldots, x_n]]$. Then $k[[x_1, \ldots, x_n]]/I$ has FRT $\iff$ $K[[x_1, \ldots, x_n]]/I$ has FRT.*

This gives a complete classification of the Gorenstein equicharacteristic local rings with FRT and with perfect residue field.

Sketch of proof of Theorem: The hardest part is to prove ascent from R to $\hat{R}$. Assume R has FRT. Show first that the Henselization R^{h} has FRT. It is enough to show that if $R \to S$ is an étale local homomorphism then S has FRT. If $R \to S$ is *finite*, this is easy: The multiplication map $S \otimes_R S \to R$ splits as $S - S$ bimodules. Therefore any MCM ${}_S N$ is a direct summand of the extended module $S \otimes_R N$ (where we view N as an R-module via restriction of scalars). It follows easily that S has FRT. (This proves ascent ($\Longrightarrow$) in part (2) as well.)

In general, push forward to get N to be a d^{th} syzygy (of, say, M) over S. Now write ${}_R M$ as a direct limit of finitely generated S-modules, use the argument above, and take syzygies over R. To do the push forward, one needs to know that R is Gorenstein on the punctured spectrum. (The original proof of this fact used Néron-Popescu desingularization and Artin approximation to show that R^{h} has a canonical module. This step can now be avoided, thanks to the recent result by Huneke and Leuschke: If R has FRT then R is an isolated singularity.)

Now, to get ascent from R^{h} to $\hat{R}$, we use the fact (Auslander, 1985) that R^{h} has an isolated singularity and hence so does $\hat{R}$, since R is excellent. Now Elkik's theorem (1974) says that every MCM $\hat{R}$-module is actually extended from R^{h}, and it follows that $\hat{R}$ has FRT.

We do not know whether the theorem is true without the hypothesis that R is excellent.

Before proving descent, we set up a little machine.

§3. The positive normal monoid add(M).

What we do now works for arbitrary finitely generated modules over any local ring R. Fix ${}_RM$, and let add(M) be the set of isomorphism classes of modules N such that $N \oplus N' \cong M^{(n)}$ for some ${}_RN'$ and some positive integer n. We view add(M) as an additive monoid, with addition given by $\oplus$.

Write $\hat{M} = V_1^{(a_1)} \oplus \cdots \oplus V_t^{(a_t)}$, where the V_i are pairwise non-isomorphic indecomposable $\hat{R}$-modules and the a_i are positive integers.

There is an obvious homomorphism $\lambda : \mathrm{add}(M) \to \mathbb{N}^t$ taking N to $(b_1, \ldots, b_t)$, where $\hat{N} = V_1^{(b_1)} \oplus \cdots \oplus V_t^{(b_t)}$. This is an embedding of monoids, and in fact it is a *divisor homomorphism*:

$$\lambda(N_1) \leq \lambda(N_2) \iff N_1 \text{ is a direct summand of } N_2.$$

Thus add(M) is identified with a *full* submonoid of $\mathbb{N}^t$, that is, $\mathrm{add}(M) \cong G \cap \mathbb{N}^t$ for some subgroup $G \leq \mathbb{Z}^{(t)}$.

(Monoids that can be representd in this form, as $G \cap \mathbb{N}^t$ for some $G \leq \mathbb{Z}^{(t)}$, are called are called *positive normal monoids.*) In particular, $\mathrm{add}(M)$ contains only finitely many indecomposables (the elements corresponding to the fundamental (minimal non-zero) elements of this monoid (the *Hilbert basis* of the monoid).

Proof of descent of FRT Assuming $\hat{R}$ has FRT, let $V_1, \ldots V_t$ be all of the indecomposable MCM $\hat{R}$-modules that actually occur in decompositions of extended MCM modules. Choose any M such that $\hat{M} \cong V_1^{(a_1)} \oplus \cdots \oplus V_t^{(a_t)}$ with each a_i positive. Then the indecomposable MCM R-modules are exactly the minimal non-zero elements of $\mathrm{add}(M)$. (This also proves descent in part (2) of the Theorem in §2.)

§4. **The number of indecomposable factors.** How badly can Krull-Schmidt fail for finitely generated modules over a local ring $(R, \mathfrak{m}, k)$? We can get some answers by examining the positive normal monoids $\mathrm{add}(M)$.

Of course we have direct-sum cancellation. Moreover, certain other kinds of failure of Krull-Schmidt are easily ruled out. For example:

Proposition. *Let M and N be indecomposable R-modules such that $M^{(r)} \cong N^{(s)}$ with r and s positive integers. Then $M \cong N$ (and $r = s$).*

Proof. We can assume $r \geq s$. Then (working in $\mathbb{N}^t$) we have $\lambda(M) \leq \lambda(N)$. Therefore M is a direct summand of N.

To get more subtle information, let $a_n(M)$ be the number of non-isomorphic indecomposable modules that are direct summands of $M^{(n)}$. (Thus $a_n(M) = 1$ for all n if and only if M is a *completely fundamental* element (Stanley) of add(M).) Since we already know that add(M) has only finitely many indecomposables, the sequence $a_*(M)$ is eventually constant, and of course it is non-decreasing. What else? Here are some specific questions:

Question 1. *Are there examples in which $a_*(M)$ is not constant?*

Question 2. *Can* $(1, 2, 2, 2, \dots)$ *occur?*

Question 3. *Can* $(1, 3, 3, 4, 4, 4 \dots)$ *occur?*

Question 4. *Can* $(1, 3, 5, 6, 6, 7, 7, 7, \dots)$ *occur?*

Question 5. *Can* $(1, \dots, 1, 3, 3, 3, \dots)$ *occur (with $a_n = 1$ for $n \leq 82$ and $a_n = 3$ for $n \geq 83$)?*

Answer to **Question 2**: No. Proof: Say $M^{(2)} \cong M(a) \oplus N^{(b)}$, where M and N are non-isomorphic indecomposables and $a \geq 0, b > 0$. Cancel copies of M until either the Proposition is violated or we have $0 = M^{(c)} \oplus N^{(b)}$, contradiction.

Answer to **Question 3**: No (Hassler, 2003).

Here is an example of a positive normal monoid for which the sequence in **Question 4** actually arises:

Let $\Phi : \mathbb{Q}^{(4)} \to \mathbb{Q}$ be given by the matrix $[1 \quad 2 \quad 3 \quad -6]$, and put $\Lambda := \mathrm{Ker}(\Phi) \cap \mathbb{N}4$. The Hilbert basis for Λ has seven elements:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}6\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\\1\end{bmatrix}, \begin{bmatrix}4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\\1\end{bmatrix}.$$

If we could represent this monoid in the form add(M), with M corresponding to the column $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, we would have $a_*(M) = (1, 3, 5, 6, 6, 7, 7, 7, \ldots)$.

Similarly, for the example in Question 5, we could take $\Phi = [\,1 \quad 82 \quad -83\,]$. The Hilbert basis for the kernel is

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\83\\82\end{bmatrix}, \begin{bmatrix}83\\0\\1\end{bmatrix}.$$

It is known (Hochster) that every positive normal monoid is isomorphic to a *Diophantine* monoid, that is, one of the form $\operatorname{Ker}(\Phi) \cap \mathbb{N}^t$ for some $s \times t$ matrix Φ with integer entries. Moreover, we have the following realization theorem:

Theorem. (______, 2001) *Let s be a fixed positive integer. Then there exists an analytically unramified one-dimensional local domain R with the following property: Let Λ be any Diophantine monoid defined by s homogeneous linear equations (i.e., $\Lambda = \operatorname{Ker}(\Phi) \cap \mathbb{N}^t$ for some t and some $s \times t$ integer matrix Φ). Assume that Λ contains* *an element* $\alpha = [a_1\ a_2\ \ldots\ a_t]^{\mathrm{t}}$ *with each* $a_i > 0$. *Then there is a MCM (= torsion-free) R-module M such that* $\mathrm{add}(M) \cong \Lambda$ *(via the embedding* λ *described earlier), and with* $\lambda([M]) = \alpha$.

In fact (Facchini and ______, 2003), one can take R to be any domain (one-dimensional and analytically unramified) whose completion contains at least $s+1$ maximal ideals, and such that R does not have FRT. (Note: Failure of FRT is automatic if $s \geq 3$).

Thus, if R is any one-dimensional local domain without FRT such that $\hat{R}$ is reduced but not a domain, then the sequences $(1,3,5,6,7,7,7,\ldots)$ and $(1,\ldots,1,3,3,3,\ldots)$ actually occur for suitable MCM R-modules M.

§**5. BRT and Brauer-Thrall.** What are the CM local rings with BRT but not FRT (counterexamples to the first Brauer-Thrall conjecture, in the context of MCM modules)?

We begin with dimension one:

Theorem. Leuschke and ______, 2003) *Let* $(R, \mathfrak{m}, k)$ *be an equicharacteristic one-dimensional local CM ring with* k *infinite. Then* R *has bounded but infinite representation type if and only if* $\hat{R}$ *is isomorphic to one of the following:*

(1) $k[[x,y]]/(y^2)$ *(the* A_∞) *singularity), or*
(2) $k[[x,y]]/(xy^2)$ *(the* D_∞ *singularity), or*
(3) *the endomorphism ring of the maximal of the ring in (2), equivalently, the ring* $k[[x,y,z]]/I$, *where* I *is the ideal of* $2{\times}2$ *minors of the matrix* $\begin{bmatrix} x & y & z \\ y & z & z \end{bmatrix}$.

For higher-dimensional hypersurfaces, we have the following:

Theorem. (Leuschke and ______, 2003) *Let* R *be a hypersurface ring* $k[[x_0, \ldots, x_d]]/(f)$, *where* k *is any infinite field. Then* R *has bounded but infinite representation type if and only if* R *is isomorphic to either*

(1) $k[[x, y, x_2, \ldots, x_d]]/(y^2 + x_2^2 + \cdots + x_d^2)$ *or*
(2) $k[[x, y, x_2, \ldots, x_d]]/(xy^2 + x_2^2 + \cdots + x_d^2)$.

Finally, a Brauer-Thrall theorem:

Theorem. (Leuschke and ______, 2003) *Let* $(R, \mathfrak{m}, k)$ *be an excellent equicharacteristic CM local ring with* k *perfect. Then* R *has finite CM type if and only if* R *has bounded CM type and* R_P *is regular for each prime ideal* $P \neq \mathfrak{m}$.

Excellence is used only in the "if" direction, and it cannot be removed.