JOHN GREENLEES (p1) 02-03-03

Chark = 2

BRAVE NEW COMMUTATIVE ALGEBRA REBORN

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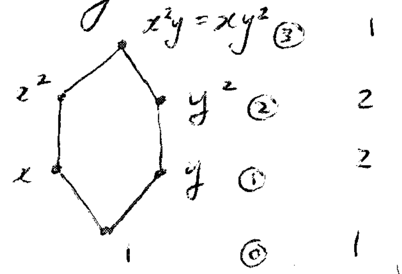
Duality previsited

1. Commutative Algebra (Gorenstein duality & local cohomology)

Example $k[x, y] = A_1$

$x^3, y^3, x^2 + xy + y^2$




$p(t) = 1 + 2t + 2t^2 + t^3$

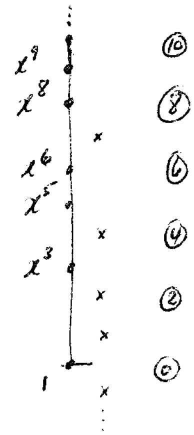
Example

$A_2 = k[x^3, x^5] \subseteq k[x]$

$$0 \to A_2 \to A_2[1/x^3] \longrightarrow \Sigma^7 A_2^\vee$$

$A_2[1/x^3] = k[x, x^{-1}]$

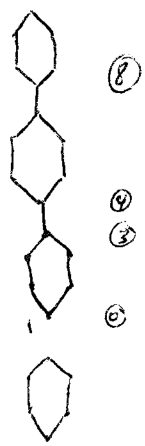
$A^\vee = \mathrm{Hom}_k(A, k)$




$$p(t) = \frac{1 - t + t^3 - t^4 + t^6 - t^7}{1 - t}$$

Example

$A_3 = A_1[z]$

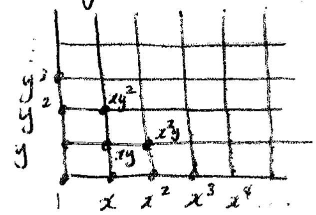



$$0 \to A_3 \longrightarrow A_3[1/z] \to \Sigma^{-1} A_3^\vee$$

$$p(t) = \frac{1 + 2t + 2t^2 + t^3}{1 - t^4}$$

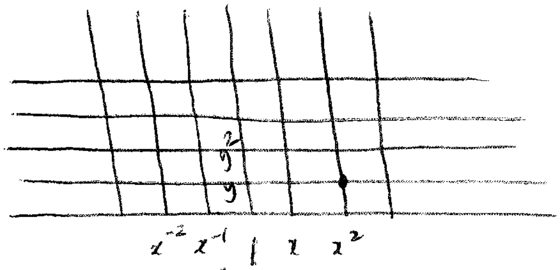
(JG p2)

<u>Example</u> $A_4 = k[\overset{①}{x}, \overset{①}{y}]$




$A_4$

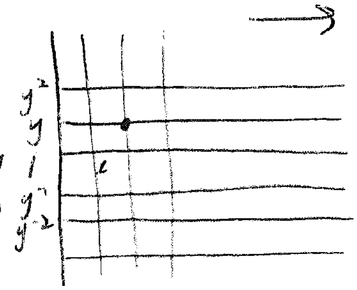
$\longrightarrow$



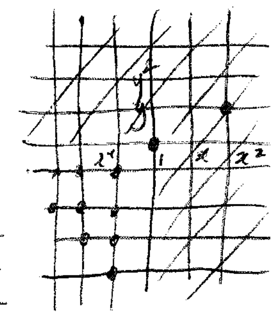

$A_4[1/x]$

$\oplus$

$A_4[1/y]$




$\longrightarrow$




$A_4[1/xy]$

$$\circled{0} \qquad \left(A_4 \to \begin{matrix} A_4[1/x] \\ \oplus \\ A_4[1/y] \end{matrix} \to A_4[1/xy]\right) \circled{0} \qquad \Sigma^{-2} A_4^\vee$$

$$\| $$

$$(A_4 \to A_4[1/x]) \otimes_{A_4} (A_4 \to A_4[1/y])$$

<u>Def^n</u>

The stable Koszul complex of $A$ with maximal ideal $I = (x_1, \ldots, x_r)$ is $K^\infty(A) = (A \to A[1/x_1]) \otimes_A \cdots \otimes_A (A \to A[1/x_r])$

The <u>local cohomology</u> is $H^*_m(A) = H^*(K^\infty(A))$

We say $A$ has <u>Gorenstein duality</u> if $H^*_m(A) = H^r_m(A) = \Sigma^? A^\vee$
($r$ = Krull dim $A$)

2. <u>Representation Theory</u>

$G$ finite 2-gp, $kG$ group ring

(JG p3)

<u>Example</u> $G = C_2 = \langle g \rangle$

$$0 \leftarrow \boxed{k} \leftarrow kG \xleftarrow{1-g} kG \xleftarrow{1+g} kG \xleftarrow{1-g} kG \xleftarrow{1+g} \cdots$$

$\downarrow \operatorname{Hom}_{kG}(\ ,k)$

$$k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \xrightarrow{2} k$$

$\downarrow H^*$

$$\operatorname{Ext}^*_{kG}(k,k) = H^*(BG) = k[\overset{1}{x}]$$

<u>Example</u> $G = \underbrace{C_2 \times C_2 \times \cdots \times C_2}_{r}$ $\qquad H^*(BG) = \operatorname{Ext}^*_{kG}(k,k) = k[\overset{1}{x_1}, \ldots, \overset{1}{x_r}]$

<u>Example</u> $G = Q_8$

$H^* BG = A_3$

## 3. <u>Interactions I: statement</u>

<u>Theorem</u> (Benson, Carlson, Lyubeznik) The cohomology ring $H^*(BG) = A$ always has a Gor. property; in fact, if $A$ is CM ($H^i_{\mathfrak{m}}(BG) = 0$ for $i < \operatorname{Kdim} A$) then $A$ is Gorenstein $p(1/t) = (-t)^r p(t)$;

if $A$ is almost CM ($H^i_{\mathfrak{m}}(A) = 0$ for $i < \operatorname{Kdim} A - 1$) then $A$ is almost Gorenstein and $p(1/t) - (-t)^r p(t) = (-1)^{r-1}(1+t)q(t)$ and $q(1/t) = (-t)^{1-r} q(t)$.

And in general there is a spectral seq$^{\underline{n}}$ $H^*_{\mathfrak{m}}(A) \Rightarrow A^\vee$

e.g. if $A$ is CM $H^r_{\mathfrak{m}}(A) \cong \Sigma^{-r} A^\vee$ and if $A$ is almost CM

$$0 \to \Sigma^{1-r} H^{r-1}_{\mathfrak{m}}(A) \to A^\vee \to \Sigma^{-r} H^r_{\mathfrak{m}}(A) \to 0$$

<u>Example</u> $G = SD_{16}$

$$H^*(BG) = \frac{k[z,y,x,w]}{(zy+y^2,\ y^3,\ yx,\ z^3x + z^2y + y^2w + x^2)} \qquad p(t) = \frac{1}{(1-t)^2(1+t^2)} \qquad r = 2, \text{ almost CM}$$

(JG p4)

4. <u>Interactions II: proof</u>

Apply brave new commutative algebra

$A = H^*BG$

Take $R = C^*(BG)$ and $K^\infty(R)$, $\underset{\displaystyle (x_1, \ldots, x_r)}{\mathfrak{m} \overset{\shortparallel}{}} =$ max$^l$ ideal of $A$ (i.e., positive dim elts)

$$K^\infty(R) = (R \to R[1/x_1]) \otimes_R \cdots \otimes_R (R \to R[1/x_r])$$

by construction

$$H^r_{\mathfrak{m}}(\underbrace{H^*R}_{A}) \Rightarrow H^*(K^\infty(R)) \qquad R^\vee = \mathrm{Hom}_k(R,k) = C_*BG \quad H_*R^\vee = H_*BG$$

Sufficient to prove $K^\infty(R) \overset{(*)}{\simeq} R^\vee$

$$\mathrm{Hom}_R(k, K^\infty(R)) \overset{①}{\simeq} \mathrm{Hom}_R(k,R) \overset{②}{\simeq} \mathrm{Hom}_R(k, R^\vee)$$

Step 3: deduce from this that $(*)$ holds

$$= \mathrm{Hom}_R(k, \mathrm{Hom}_k(R,k)) = \mathrm{Hom}_R(R \otimes_k k, k) = k$$

Step 1: $\mathrm{Hom}_R(k, K^\infty(R)) \simeq \mathrm{Hom}_R(k,R)$

All terms in $K^\infty(R)$ except $R \otimes_R \cdots \otimes_R R = R$ invert some $x \in \mathfrak{m}$.

$\mathrm{Hom}_R(\underset{R/\mathfrak{m}}{k}, R[1/x]) \simeq 0$

Step 2: $R = C^*(BG)$, $\mathcal{E} = kG$ $\quad \mathrm{Hom}_{\mathcal{E}}(k,k) = C^*BG$, $k \otimes_{\mathcal{E}} k = C_*BG = R^\vee$

$\mathrm{Hom}_R(k,k) = C_*\Omega BG = \mathcal{E}$, $k \otimes_R k = \mathcal{E}^\vee$ (Eilenberg-Moore)

$\mathrm{Hom}_R(k,R) \simeq k$ is what we want.

$$\mathrm{Hom}_R(k, R^{\vee\vee}) = \mathrm{Hom}_R(R^\vee \otimes_R k, k) = \mathrm{Hom}_k(k \otimes_{\mathcal{E}} \underbrace{k \otimes_R k}, k) = \mathrm{Hom}_k(k \otimes_{\mathcal{E}} \mathcal{E}^\vee, k) = \mathrm{Hom}_{\mathcal{E}}(k, \mathcal{E}^\vee) = k$$

Step 3 $\mathrm{Hom}_R(k, H) \qquad H = K^\infty(R)$ or $R^\vee$

$$\underbrace{\mathrm{Hom}_R(k,H)}_{\text{right } \mathcal{E}\text{-module}} \otimes_{\mathcal{E}} k \xrightarrow{\simeq} H$$

(Obvious for $H = k$, $\therefore$ for any $H$ built from $k$, e.g. $H = K^\infty(R)$ or $R^\vee$)