Resolutions with Structure

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Commutative Algebra: Interactions with Homological Algebra and Representation Theory

MSRI
February 2003

NOTATION:

K a field

$S = K < x_1, \ldots, x_n >$ or $S = KQ$.

$S = S_0 \oplus S_1 \oplus S_2 \cdots$ with

$S_0 = \prod_{i=1}^{n} K$ and S_i finite dim'l over K

Here S_i is the K-span of the monomials of degree i or the paths of length i.

S generated in degrees $0, 1$.

$J = S_1 \oplus S_2 \oplus S_3 \oplus \cdots$

So $J =< x_1, \ldots, x_n >$ or $J =<$ arrows of $Q >$

I an ideal in S with $I \subseteq J^2$.

$R = S/I$ and $\mathbf{r} = J/I$.

Note that S has a Gröbner basis theory: that is,

1. S has a multiplicative K-basis, $\mathcal{B}$; i.e., if $b_1, b_2 \in \mathcal{B}$ then $b_1 b_2 \in \mathcal{B}$ or $b_1 b_2 = 0$.

2. There is an *admissible* order on $\mathcal{B}$; i.e.,

   (a) $>$ is a well-order.

   (b) if $b_1, b_2, b_3 \in \mathcal{B}$ and $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$ if both not 0 and
   $b_3 b_1 > b_3 b_2$ if both not 0

   (c) if $b_1, b_2, b_3 \in \mathcal{B}$ and $b_1 = b_2 b_3$ then $b_1 \geq b_2$ and $b_1 \geq b_3$.

We call $(\mathcal{B}, >)$ an *ordered multiplicative basis*.

1. $S = K < x_1, \ldots, x_n >$ with $\mathcal{B} = \{\text{monomials}\}$. Thus
   $R = K < x_1, \ldots, x_n > /I$ with
   $I \subseteq < x_1, \ldots, x_n >^2$.

2. $S = KQ$ with $\mathcal{B} = \{\text{paths}\}$

   Note that if M is an S_0-S_0 bimodule then, the tensor algebra

   $S = T_{S_0}(M) = S_0 \oplus \ M \oplus \ (M \otimes_{S_0} M) \oplus \cdots$

   is a path algebra.

   Thus $K < x_1, \ldots, x_n >$ is a path algebra.

Graded algebras

If I can be generated by homogeneous elements of S, then R has an induced grading from S.

In this case, write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$.

Note that $R_0 = \prod_{i=1}^{n} K$ and
$\mathbf{r} = R_1 \oplus R_2 \oplus \cdots$.

Graded R-modules have graded projective resolutions.

Whether or not R is graded, we will denote $R/\mathbf{r} = S/J = \prod_{i=1}^{n} K$ by R_0.

$E(R) = \oplus_{m \geq 0} \mathrm{Ext}_R^m(R_0, R_0)$, a ring via the Yoneda product.

If M is an R-module,
$E(M) = \oplus_{m \geq 0} \mathrm{Ext}_R^m(M, R_0)$.

$E(M)$ is naturally an $E(R)$-module.

Review of Koszul Algebras

Assume that I is generated by homogeneous elements.

$R = S/I$ is *Koszul* if R_0 has a linear (graded) projective resolution:

$$\cdots \to P^2 \to P^1 \to P^0 \to R_0 \to 0$$

P^n generated in degree n.

Properties

1. I is generated in degree 2.

2. $E(R) = \mathrm{Ext}^*_R(R_0, R_0)$ is generated in degrees 0 and 1.

3. $E(R)$ is a Koszul algebra.

4. R^{op} is a Koszul algebra.

5. The Koszul complex is exact.

6. If $R = KQ/I$, then
$E(R) = KQ^{\text{op}}/ < I_2^{\perp} >$.

7. $\text{Ext}^*(-, R_0) : \text{Mod}(R) \to \text{Mod}(E(R))$ is a duality on the category of Koszul modules.

I. D-Koszul Algebras

Joint with E. N. Marcos, Brazil, R. Martínez-Villa, Mexico, and Pu Zhang, China

Assume that I can be generated by homogeneous elements.

$R = R_0 \oplus R_1 \oplus \cdots$ is *d-Koszul* if there is a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to R_0 \to 0$$

with degree

$$P^n = \begin{cases} \frac{n-1}{2}d, & \text{if } n \text{ odd,} \\ \frac{n}{2}d, & \text{if } n \text{ even.} \end{cases}$$

Although restrictive, there are many such algebras. Introduced by Roland Berger.

Proposition 1 *If R is d-Koszul with P^2 generated in degree d, I can be generated in degree d.*

Theorem 2 *If $R = KQ/I$ and I is generated in degree d then R is d-Koszul if and only if $E(R)$ is generated in degrees $0, 1$ and 2.*

Proposition 3 *If R is d-Koszul then $E^{ev}(R) = \oplus_{n \geq 0} Ext^{2n}(R_0, R_0)$ is a Koszul algebra.*

If R is d-Koszul, then

$$\mathrm{Ext}^{odd}(R_0, R_0) \cdot \mathrm{Ext}^{odd}(R_0, R_0) = 0.$$

Regrade $E(R)$:

$$E(R)_0 = \mathrm{Ext}^0(R_0, R_0)$$

$$E(R)_1 = \mathrm{Ext}^1(R_0, R_0) \oplus \mathrm{Ext}^2(R_0, R_0)$$

$$E(R)_2 = \mathrm{Ext}^3(R_0, R_0) \oplus \mathrm{Ext}^4(R_0, R_0)$$

In general,

$$E(R)_n = \mathrm{Ext}^{2n-1}(R_0, R_0) \oplus \mathrm{Ext}^{2n}(R_0, R_0)$$

Theorem 4 *If R is d-Koszul then $E(R)$ (re-graded) is a Koszul algebra.*

Proposition 5 *If R is d-Koszul then R^{op} is d-Koszul.*

There is a generalized Koszul complex:

Let $V = R_1$ so that $KQ = T_R(V)$ where $R = \prod_{i=1}^n K$.

We let V^a denote $\otimes_{R_0}^a V$.

$G =$ span of a set of generators of I.

$G \subset V^d$

Let $S^0 = R_0$, $S^1 = V$.

For $n \geq 2$,

$$S^n = \begin{cases} \sum_i V^i \otimes_{R_0} G \otimes_{R_0} V^{(dn/2)-d-i}, & \text{if } n \text{ even} \\ \sum_i V^i \otimes_{R_0} G \otimes_{R_0} V^{(d(n-1)/2)-d-i+1}, & \text{if } n \text{ odd} \end{cases}$$

Note that $S^n \subset V^{dn/2}$ or $S^n \subset V^{d(n-1)/2+1}$.

$$Q^n = R \otimes_{R_0} S^n.$$

There is a natural $d^n : Q^n \to Q^{n-1}$

$$d^n(\sum \lambda \otimes [v_1 \otimes \cdots \otimes v_{dn/2}]) =$$

$$\sum(\lambda v_1) \otimes [v_2 \otimes \cdots \otimes v_{dn/2}]$$

or

$$d^n(\sum \lambda \otimes [v_1 \otimes \cdots \otimes v_{d(n-1)/2+1}]) =$$

$$\sum(\lambda(v_1 \otimes \cdots \otimes v_{d-1})) \otimes [v_d \otimes \cdots \otimes v_{d(n-1)/2+1}]$$

Proposition 6 *$(Q^\bullet, d^\bullet)$ is a complex.*

Theorem 7 *Let $R = KQ/I$ with I generated in degree d. Then R is d-Koszul if and only if $(Q^\bullet, d^\bullet)$ is a projective resolution of R_0.*

Let G be a subspace of V^d and let
$I = < G >$ in $KQ = T_R(V)$.

Then $G^{\perp} \subset \otimes^d V^*$

Consider $A = T_R(V^*)/ < G^{\perp} >$. This is a graded algebra.

Theorem 8 *(R.Berger) Keeping the above notation, if $R = KQ/ < G >$ is a d-Koszul algebra then Ext$^n(R_0, R_0)$ is isomorphic to $A_{dn/2}$ if n is even and*
$A_{d(n-1)/2+1}$ if n is odd.

We show that the "induced" algebra structure from A is, in fact, the algebra structure of $Ext^*(R_0, R_0)$.

There is a classification of monomial d-Koszul algebras.

II. δ-Koszul Algebras

Let $I \subset J^2$ be generated by elements of degree d. Then $R = R_0 \oplus R_1 \oplus \cdots$.

ASSUME that there is an admissble order such that I has a Gröbner basis consisting of elements of degree d.

Consider a minimal graded projective resolution:

$$\cdots \to P^2 \to P^1 \to P^0 \to R_0 \to 0$$

Suppose there is a function
$\delta : \mathbf{N} \to \mathbf{N}$ such that P^n is generated in degree $\delta(n)$. We say R is δ-*preKoszul*. If $E(R)$ is finitely generated, we say R is δ-*Koszul*.

R is Koszul iff $\delta(n) = n$.

R is d-Koszul iff

$$\delta(n) = \begin{cases} \frac{n-1}{2}d, & \text{if } n \text{ odd,} \\ \frac{n}{2}d, & \text{if } n \text{ even.} \end{cases}$$

Are there any other δs possible? What are they? Is $E(R)$ special for these δs?

We have $\delta(0) = 0, \delta(1) = 1$, and $\delta(2) = d$.

Suppose $0 < c < d$ and $d \equiv r \mod(c)$ with $0 < r \leq c$. Note if $c = 1$, $r = 1$.

Let $\delta_{c,d}(n)$ be defined by

1. $\delta_{c,d}(0) = 0, \delta_{c,d}(1) = 1$, and $\delta_{c,d}(2) = d$.

2. For $n \geq 3$,

$$\delta_{c,d}(n) = \{ \begin{array}{ll} \frac{n-1}{2}(d+c) - \frac{n-3}{2}r, & \text{if } n \text{ odd,} \\ \frac{n}{2}d + \frac{n-2}{2}(c-r), & \text{if } n \text{ even.} \end{array}$$

Note: if $d = 2$ (so $c = 1$ and $r = 1$) then $\delta_{1,2}(n) = n$. So we have Koszul is the same as $\delta_{1,2}$-(pre)Koszul.

If $d > 2$ and $c = 1$ and hence $r = 1$, then

$$\delta_{1,d} = \{ \begin{array}{ll} \frac{n-1}{2}d, & \text{if } n \text{ odd,} \\ \frac{n}{2}d, & \text{if } n \text{ even.} \end{array}$$

We have d-Koszul is the same as $\delta_{1,d}$-(pre)Koszul.

Let R be a d-Koszul algebra.

A module M is *d-Koszul* if there is a projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$$

such that if n is even, P^n is generated in degree $dn/2$ and if n is odd, P^n is generated in degree $d(n-1)/2 + 1$.

Theorem 9 *If R is d-Koszul and M is a d-Koszul module then*

1. *$\mathrm{Ext}^{even}(M, R_0)$ is a Koszul module over the Koszul algebra $\mathrm{Ext}^{even}(R_0, R_0)$.*

2. *$\mathrm{Ext}^*(M, R_0)$, after regrading, is a Koszul module over (the regraded) Koszul algebra $E(R)$.*

Theorem 10 *1. Let $R = KQ/I$ such that I has a Gröbner basis consisting of homogeneous elements of degree d for some admissible order. If R is δ-preKoszul then there exist c, d, $0 < c < d$ such that $\delta = \delta_{c,d}$.*

2. For each c, d, $0 < c < d$, there is a $\delta_{c,d}$-preKoszul algebra RKQ/I such that I has a Gröbner basis consisting of homogeneous elements of degree d for some admissible order.

Theorem 11 *Let* $0 < c < d$ *and* $r \equiv d$ *mod*(c)*,* $0 < r \leq c$*. Then*

1. *If* $d = 2$ *and* R *is a* $\delta_{1,2}$*-preKoszul algebra then* $E(R)$ *is generated in degrees* $0, 1$*. Hence* R *is* $\delta_{1,2}$*-Koszul.*

2. *If* $d > 1$ *and* $c = 1$ *and* R *is a* $\delta_{1,d}$*-preKoszul algebra then* $E(R)$ *is generated in degrees* $0, 1, 2$*. Hence* R *is* $\delta_{1,2}$*-Koszul.*

3. *If* $d > 1, c > 1$*, and* $r = 0$ *(i.e.,* $c \mid d$*) and* $R = KQ/I$ *a* $\delta_{c,d}$*-preKoszul algebra and* I *with a degree* d *homogeneous Gröbner basis then* $E(R)$ *is generated in degrees* $0, 1, 2, 3$*. Hence* R *is* $\delta_{1,2}$*-Koszul.*

4. *If* $d > 1, c > 1$*, and* $r \neq 0$ *and* R *is a* $\delta_{c,d}$*-preKoszul monomial algebra then* $E(R)$ *is not finitely generated. Hence* R *is not* $\delta_{c,d}$*-Koszul.*

Questions: Can there other δs if I does not have Gröbner basis of homogeneous elements of one degree? Can there be δ-Koszul algebras for new δs?

III. Quasi-Koszul Algebras and their Resolutions

joint with Yuriy Drozd, Kiev

We now look at NONgraded algebras.

We keep the same notational conventions; i.e.,
$S = K < x_1, \ldots, x_n >$ or $S = KQ$
I is an ideal in S with $I \subseteq J^2$
$R = S/I$.

Two Examples:

I: $R = K < x,y,z > /(x^2 - z^3)$

$E(R) = \text{Ext}^*_R(K,K)$ is not a Koszul algebra.

$$0 \to R \overset{\begin{pmatrix} x \\ 0 \\ -z^2 \end{pmatrix}}{\to} R^3 \overset{(x,y,z)}{\to} R \to K \to 0$$

II: $R = K < x,y > /(xy - z^3)$

$E(R)$ is a Koszul algebra

$$0 \to R \overset{\begin{pmatrix} y \\ 0 \\ -z^2 \end{pmatrix}}{\to} R^3 \overset{(x,y,z)}{\to} R \to K \to 0$$

Def: A K-algebra R is a *quasi-Koszul algebra* with respect to an ideal $\mathbf{r}$ if

1. $R/\mathbf{r}$ is $\prod_{i=1}^{n} K$.

2. $E(R) = \mathrm{Ext}_R^*(R/\mathbf{r}, R/\mathbf{r})$ is a Koszul algebra.

$\mathrm{Gr}_{\mathbf{r}}(R) = R/\mathbf{r} \oplus \mathbf{r}/\mathbf{r}^2 \oplus \mathbf{r}^2/\mathbf{r}^3 \oplus \cdots$ assoc. graded with respect to the $\mathbf{r}$-adic filtration

Def Let M be an R-module. We say a projective resolution, $(P^\bullet, d^\bullet)$, of M is *quasi-linear* if

1. each P^n is finitely generated

2. $d^n(P^n) \subseteq \mathbf{r}P^{n-1}$

3. the complex $(\mathrm{Gr}_{\mathbf{r}}(P^\bullet), \hat{d}^\bullet)$ is a projective resolution of the $\mathrm{Gr}_{\mathbf{r}}(R)$-module $\mathrm{Gr}_{\mathbf{r}}(M)$.

$\mathsf{Gr}_{\mathbf{r}}(M) = M/\mathbf{r}M \oplus \mathbf{r}M/\mathbf{r}^2M \oplus \mathbf{r}^2M/\mathbf{r}^3M \oplus \cdots$

$\hat{d}^n(x + \mathbf{r}^k P^n) = d^n(x) + \mathbf{r}^{k+1}P^{n-1}$

$((\mathbf{r}^{k-1}P^n \to \mathbf{r}^k P^{n-1} \to \mathbf{r}^k P^{n-1}/\mathbf{r}^{k+1}P^{n-1}$
induces
$\mathbf{r}^{k-1}P^n/\mathbf{r}^k P^n \to \mathbf{r}^k P^{n-1}/\mathbf{r}^{k+1}P^{n-1}))$

Theorem 12 *Suppose that there is a quasi-linear resolution of $R/\mathbf{r}$. Then*

1. *$E(R)$ is isomorphic to $E(Gr_{\mathbf{r}}(R))$. In particular, R is a quasi-Koszul algebra and $E^2(R)$ is isomorphic to $Gr_{\mathbf{r}}(R)$.*

2. *If an R-module M has a quasi-linear resolution, $E(M) = Ext^*_R(M, R/\mathbf{r})$ is a Koszul $E(R)$-module and $E^2(M)$ is isomorphic to $Gr_{\mathbf{r}}(R)$.*

If $x \in S \setminus \{0\}$, $x = x_k + \cdots + x_{k+r}$, where $x_i \in S_i$ and $x_k \neq 0$.

Set $u(x) = x_k$, $t(x) = x - x_k$.

Let $\mathcal{F} = \{f_i\}_{i \in \mathcal{I}}$ be a set of generators of I $(R = S/I)$

$H =< \{u(f_i)\} >_{i \in \mathcal{I}}$. H is a homogeneous ideal in S.

$\Gamma = S/H = \Gamma_0 \oplus \Gamma_1 \oplus \cdots$ −grading induced from the grading in S.

There is a natural surjection $\varphi : \Gamma \to \mathrm{Gr}_{\mathbf{r}}(R)$. Not an iso in general:

<u>Ex</u> $R = K < x, y, z > / < \mathcal{F} >$, $\mathcal{F} = \{xy + z^3, y^2\}$

$H =< xy, y^2 >$ and φ is not an iso.

$>$ admissible order: deg-lex with $\deg(x) = 3$, $\deg(y) = deg(z) = 1$.

$\mathcal{F}$ not a Gröbner basis of I.

CONDITION $(*)$:

For each $f \in \mathcal{F}$, $\text{Tip}(f) = \text{Tip}(u(f))$

Let $\mathcal{Q} = \{u(f)\}_{f \in \mathcal{F}}$.

Example above satisfies condition $(*)$.

Theorem 13 *Keeping the notations above, assume that $>$ is an admissible order on $\mathcal{B}$, the multiplicative basis of S. Assume that $\mathcal{F}$ satisfies condition $(*)$. Suppose that $\mathcal{F}$ is a Gröbner basis for I with respect to $>$. Then*

1. *$\mathcal{Q}$ is a Gröbner basis for H with respect to $>$.*

2. *$\varphi : S/H \to \text{Gr}_{\mathbf{r}}(R)$ is an isomorphism.*

Putting it all together:

$S = S_0 \oplus S_1 \oplus \cdots$, S has an ordered multiplicative basis $\mathcal{B}, >$ that respects the grading.

$J = S_1 \oplus S_2 \oplus \cdots$ and S generated in degrees $0, 1$.

I is an ideal generated by $\mathcal{F} \subset J^2$

Assume $\mathcal{F}$ satisfies condition $(*)$; i.e., $f \in \mathcal{F}$ implies $\text{Tip}(f) = \text{Tip}(u(f))$.

H ideal generated by $\mathcal{Q} = \{u(f)\}_{f \in \mathcal{F}}$.
$\Gamma = S/H$.

Theorem 14 *Keeping the above notation and assumptions, if $\mathcal{F}$ is a Gröbner basis for I with respect to $>$ and $\mathcal{Q}$ consist of quadratic elements, then $R/\mathbf{r}$ has quasi-linear projective resolution. In this case,*

1. R *is a quasi-Koszul algebra.*

2. $S/H \simeq Gr_{\mathbf{r}}(R)$ *is a Koszul algebra.*

3. $Ext^*_R(R/\mathbf{r}, R/\mathbf{r}) \simeq Ext^*_{Gr_{\mathbf{r}}(R)}(R/\mathbf{r}, R/\mathbf{r})$ *is a Koszul algebra.*

4. $E^2(R) \simeq Gr_{\mathbf{r}}(R)$.

Two Examples:

I: $R = K < x, y, z > /(x^2 - z^3)$

$E(R) = \text{Ext}_R^*(K, K)$ is not a Koszul algebra.

$$0 \to R \overset{\begin{pmatrix} x \\ 0 \\ -z^2 \end{pmatrix}}{\to} R^3 \overset{(x,y,z)}{\to} R \to K \to 0$$

$\mathcal{F} = \{x^2 - z^3\}$ is NOT a Gröbner basis for $< x^2 - z^3 >$ for any order.

II: $R = K < x, y > /(xy - z^3)$

$E(R)$ is a Koszul algebra

$$0 \to R \overset{\begin{pmatrix} y \\ 0 \\ -z^2 \end{pmatrix}}{\to} R^3 \overset{(x,y,z)}{\to} R \to K \to 0$$

Take $>$ to be deg-lex with $\deg(x) = 3$, $\deg(y) = \deg(z) = 1$.

$\mathcal{F} = \{xy - z^3\}$ IS a Gröbner basis for $< xy - z^3 >$ and $\mathcal{Q} = \{xy\}$ is a quadratic Gröbner basis for $\mathrm{Gr}_{\mathbf{r}}(R)$.

QUESTIONS:

1. If $\mathcal{F}$ satisfies $(*)$ but is not a Gröbner basis for I, can $\varphi : S/H \to \mathrm{Gr}_{\mathbf{r}}(R)$ be an isomorphism?

2. Can there be quasi-Koszul algebras (i.e., $E(R)$ Koszul) but $R/\mathbf{r}$ does not have a quasi-linear projective resolution?