Depth, Detection, Something and Something
(I can't remember)

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G = a finite group (a p-group)

k = a field of characteristic p

The cohomology ring $\mathrm{H}^*(G, k)$ is a finitely generated graded commutative k-algebra.

Problems

- How are the properties of the group reflected in the structure of the cohomology ring?
- How are cohomology rings special?

A lot is known

- (Quillen) Let $V_G(k)$ denote the maximal ideal spectrum of $\mathrm{H}^*(G,k)$. Then

  $$V_G(k) \;=\; \cup_E \mathrm{res}^*_{G,E}(V_E(k))$$

  where E runs through the maximal elementary abelian p-subgroups of G and

  $$\mathrm{res}_{G,E} : \mathrm{H}^*(G,k) \longrightarrow \mathrm{H}^*(E,k)$$

  is the restriction map.
- (Duflot) The depth of $\mathrm{H}^*(G,k)$ is at least equal to the p-rank of the center of a Sylow p-subgroup of G.

Examples

Suppose that $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ is an elementary abelian group of order p^n. Then

$$\mathrm{H}^*(G,k) \cong \begin{cases} k[\zeta_1,\dots,\zeta_n] \otimes \Lambda(\eta_1,\dots,\eta_n) & \text{if } p > 2 \\ k[\zeta_1,\dots,\zeta_n] & \text{if } p = 2 \end{cases}$$

where $deg(\zeta_i) = 2$, $deg(\eta_i) = 1$ when $p > 2$ and $deg(\zeta_i) = 1$ when $p = 2$.

Suppose that $p = 2$ and G is a dihedral group of order 8. Then

$$\mathrm{H}^*(G,k) \;\cong\; k[z,y,x]/(zy)$$

where $deg(z) = deg(y) = 1$ and $deg(x) = 2$.

Suppose that $p = 2$ and G is the group of order 128 generated by $g_1, \ldots, g_7$ with relations

$$
\begin{array}{lll}
g_1^2 = g_5g_6g_7, & g_2^2 = g_4, & g_3^2 = g_6g_7, \\
g_5^2 = g_7, & g_2^{g_1} = g_2g_3, & g_3^{g_2} = g_3g_5, \\
g_4^{g_1} = g_4g_5g_6g_7, & g_4^{g_3} = g_4g_6g_7, & g_5^{g_2} = g_5g_6, \\
g_5^{g_4} = g_5g_7, & g_6^{g_2} = g_6g_7. &
\end{array}
$$

Then $\mathrm{H}^*(G, k) \cong k[z, y, x, w, v, u, t, s]/\mathcal{I}$ where the degrees of the variables are $1, 1, 2, 2, 2, 3, 3, 4$ and the ideal $\mathcal{I}$ is generated by the elements

$$
\begin{gathered}
zy, \quad y^2, \quad z^3 + yx + zv, \quad zx + yx, \quad yw, \quad z^2w + zt, \\
x^2 + xv, \quad yu, \quad yt, \quad yxv + xu + xt, \quad z^2u + xu + vu, \\
xw^2 + zwu + ut, \quad w^2v + t^2, \quad zvt + z^2s + u^2 + ut.
\end{gathered}
$$

Detection

Let $\mathcal{H}$ be a collection of subgroups of G. We say that the cohomology of G is detected on $\mathcal{H}$ if the map

$$\prod_{H\in\mathcal{H}} \mathrm{res}_{G,H} : \quad \mathrm{H}^*(G,k) \longrightarrow \prod_{H\in\mathcal{H}} \mathrm{H}^*(H,k)$$

is injective.
In other words, the intersection of the kernels of the restriction maps to the elements of $\mathcal{H}$ is the zero ideal.

Theorem 1. *Suppose that the cohomology ring $H^*(G,k)$ has depth d. Then the cohomology is detected on the centralizers of the elementary abelian p-subgroups of rank d.*

Let

$d =$ depth of $\mathrm{H}^*(G,k)$.

$s =$ least t such that $\mathrm{H}^*(G,k)$ is detected on the centralizers of the elementary abelian p-subgroups of rank t.

$a =$ least t such that $\mathrm{H}^*(G,k)$ has an associated prime $\mathfrak{p}$ with $\dim \mathrm{H}^*(G,k)/\mathfrak{p} = t$.

We know that

$$d \leq s \leq a$$

Question: Is $d = a$ always?

Example

Let $R \subseteq k[z, y]$ be the subalgebra generated by the elements

$$1, \quad z^2, \quad z^3, \quad zy, \quad y$$

Alternatively,

$$R \cong k[a, b, c, d]/(a^3 - b^2, ac - bd, c^2 - ad^2)$$

Then R is an integral domain with depth 1 and Krull dimension 2.

Question: What can we know about rings in which the depth is determined by the associated primes.

Isomorphisms

Let $p = 2$, $k = \mathbb{F}_2$, and let

$$D_{2^n} = \langle u, v | u^2 = v^2 = (uv)^{2^{n-1}} = 1 \rangle.$$

Then

$$\mathrm{H}^*(D_{2^n}, k) \cong k[z, y, x]/(zy)$$

where the degrees of z, y and x are $1, 1$ and 2.

So we have

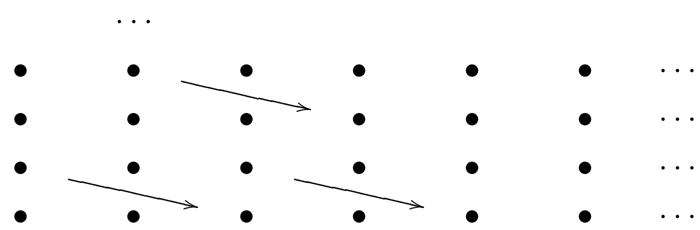
$$1 \longrightarrow C_2 \longrightarrow D_{16} \longrightarrow D_8 \longrightarrow 1$$

and an inflation map

$$\inf : \mathrm{H}^*(D_8, k) \longrightarrow \mathrm{H}^*(D_{16}, k).$$

But this is not an isomorphism!

The spectral sequence



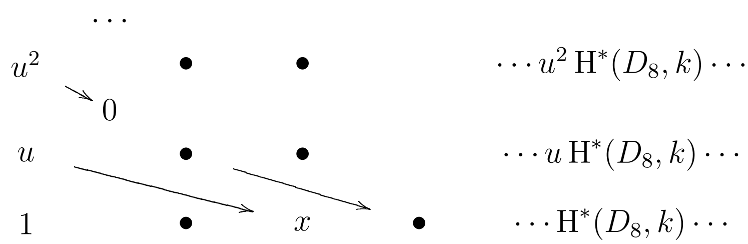
$$E_2^{r,s} \cong \mathrm{H}^r(D_8, \mathrm{H}^s(C_2, k)) \cong \mathrm{H}^r(D_8, k) \otimes \mathrm{H}^s(C_2, k) \;\Rightarrow\; \mathrm{H}^{r+s}(D_{16}, k)$$

$$d_2 : E_2^{r,s} \longrightarrow E_2^{r+2,s-1}$$

Note that $\mathrm{H}^s(C_2, k) \cong k$ for all $s \geq 0$.

Remember $\mathrm{H}^*(D_8, k) \cong k[z, y, x]/(zy)$ and $\mathrm{H}^*(C_2, k) \cong k[u]$

So the spectral sequence looks like



That is, d_2 on the odd rows is multiplication by x and is 0 on the even rows.

So the E_3 page of the spectral sequence looks like

$$
\begin{array}{ccccc}
 & \cdots & & & \\
u^4 & \bullet & \bullet & & \cdots u^4\,\mathrm{H}^*(D_8,k)/(x)\cdots \\
0 & 0 & 0 & 0 & \\
u^2 & \bullet & \bullet & & \cdots u^2\,\mathrm{H}^*(D_8,k)/(x)\cdots \\
0 & 0 & 0 & 0 & \\
1 & \bullet & \bullet & \bullet & \cdots \mathrm{H}^*(D_8,k)/(x)
\end{array}
$$

Consequently, $\mathrm{H}^*(D_{16},k) \cong k[z,y,u^2]/(zy)$

General Method

Suppose that $\mathrm{H}^2(G,k)$ ($k=\mathbb{F}_p$) has an element ζ such that

$$\mathrm{H}^*(G,k) \cong A \otimes_k k[\zeta]$$

for A some subalgebra, and

$$\beta(\zeta) \in \zeta\, \mathrm{H}^*(G,k)$$

where β is the Bockstein homomorphism. Then there is an extension

$$1 \longrightarrow C_p \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

such that $\mathrm{H}^*(\hat{G},k) \cong \mathrm{H}^*(G,k)$.

Some extensions of this type were investigated by Browder-Pakianathan, Weigel. But there are others.

Let $P = k[z, y, x, w, v, u, t, s]$ where the variables are in degrees $1, 1, 2, 2, 2, 2, 3, 3, 4$. The for a particular family the cohomology ring has the form $\mathrm{H}^*(G, k) \cong P/\mathcal{I}$ where

a. For the group of order 256, $\mathcal{I} := \langle\, z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

b. For the group of order 128, $\mathcal{I} := \langle\, z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

a. For the group of order 64, $\mathcal{I} := \langle\, z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

a. For the group of order 32, $\mathcal{I} := \langle\, z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

The images of the generators of the cohomology of 128 inflated to 256 are $z, y, x, w, 0, u, t, s$

The images of the generators of the cohomology of 64 inflated to 128 are $z, y, x, w, 0, u, t, s$

The images of the generators of the cohomology of 32 inflated to 64 are $z, y, x, w, 0, u, t, s$

Let $P = k[z, y, x, w, v, u, t, s]$ where the variables are in degrees $1, 1, 2, 2, 2, 2, 3, 3, 4$. The for a particular family the cohomology ring has the form $\mathrm{H}^*(G, k) \cong P/\mathcal{I}$ where

a. For the group of order 256,

$$zy, \quad y^2, \quad z^3 + yx, \quad zx + yx, \quad yw, \quad z^2v + zt,$$
$$x^2, \quad yu, \quad yt, \quad z^2u + z^2t + xt, \quad xu,$$
$$z^2w^2 + zvt + t^2, \quad xw^2 + zvu + zvt + ut + t^2,$$
$$zwu + zvu + zvt + z^2s + u^2 + ut + t^2.$$

b. For the group of order 128,

$$zy, \quad y^2, \quad z^3 + yx, \quad zx + yx, \quad yw, \quad z^2w + zt,$$
$$x^2, \quad yu, \quad yt, \quad z^2u + xt, \quad xu,$$
$$xw^2 + zwu + zwt + ut, \quad z^2s + u^2 + ut, \quad t^2.$$

The two cohomology rings are isomorphic – but we must make the change of variables

$$t \mapsto t + zw + zv$$

and

$$s \mapsto s + w^2$$

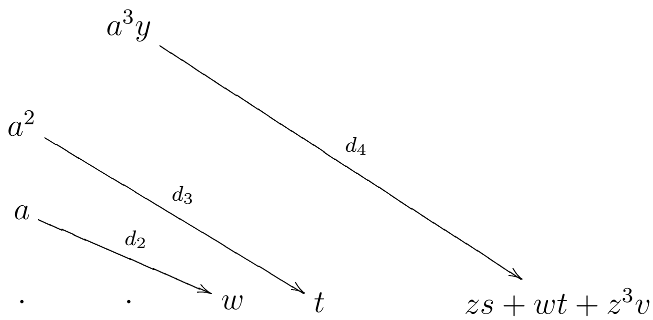
The images of the generators of the cohomology of 126 inflated to 256 are

$$z, \quad y, \quad x, \quad 0, \quad v, \quad zw, \quad 0, \quad z^2v + w^2.$$

E_2 page of spectral sequence

a^3	a^3z, a^3y	$\cdots$		
a^2	a^2z, a^2y	$\cdots$		
a	az, ay	ax, aw, av	$\cdots$	
1	z, y	x, w, v	u, t	s

As to the differentials, we have



So the distribution of the new generators (for the group of order 256) is

s'		
	u', t'	
	w'	
1	z', y'	x', v'

$(ay \mapsto w', \quad a^2 z \mapsto u', \quad a^2 y \mapsto t' \quad \text{and } a^4 \mapsto s'.)$

JOHN CARLSON (blackboard notes)

$R/(x_1, \ldots, x_i) \xrightarrow{x_{i+1}} R/(x_1, \ldots, x_i)$ is injective in degrees $\geq \deg x_1 + \ldots + \deg x_i$

$x_1, \ldots, x_n$ quasi-regular