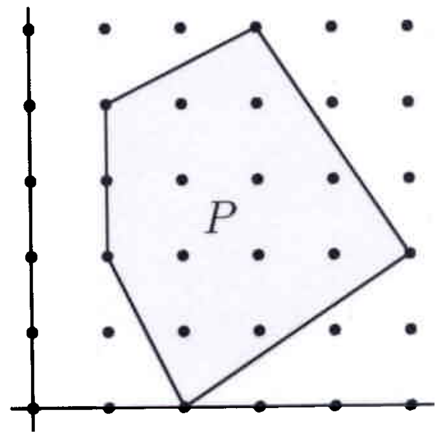
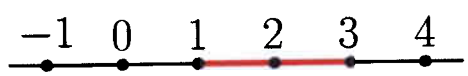
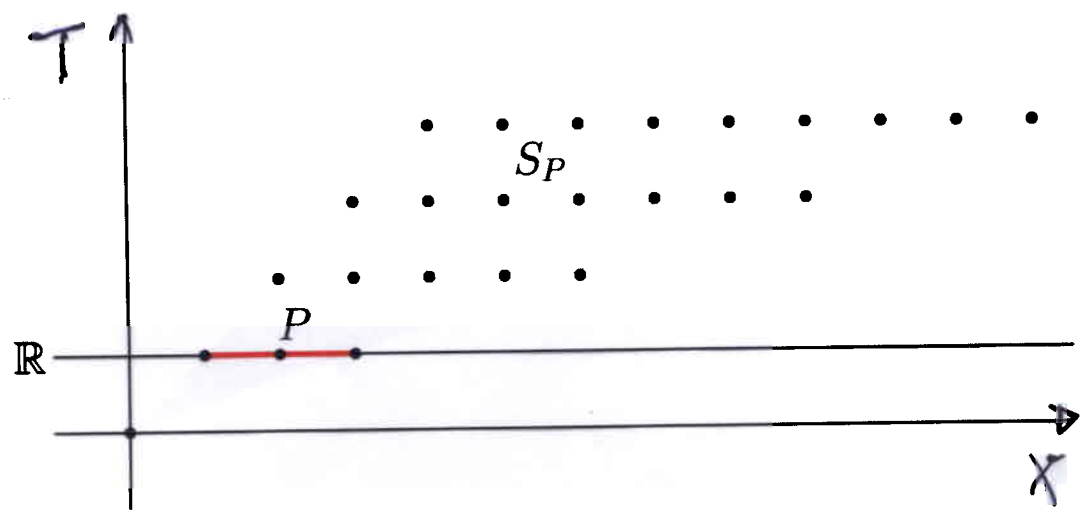


A lattice polytope



$$P = [1,3] \subset \mathbb{R}, \qquad L_P = \{1,2,3\}$$



$$K[S_P] = K[XT, X^2T, X^3T] \cong K[Y^2, YZ, Z^2]$$

$$\cong K[u,v,w]/(uw - v^2)$$

Milnor's K_2

(i) $E_n \longrightarrow E_{n+1}$ $\qquad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$\boxed{E(R) = \varinjlim E_n}$

(ii) Steinberg relations

$$e_{ij}^{\lambda}\, e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$$

$$[e_{ij}^{\lambda}, e_{jk}^{\mu}] = e_{ik}^{\lambda\mu} \qquad i \neq k$$

$$[e_{ij}^{\lambda}, e_{ki}^{\mu}] = e_{kj}^{-\lambda\mu} \qquad j \neq k$$

$$[e_{ij}^{\lambda}, e_{kl}] = 1 \qquad i \neq l,\ j \neq k$$

$$St(R) = \langle x_{ij}^{\lambda} : i,j \geq 1,\ i \neq j,\ \lambda \in R \rangle / \text{Steinberg relations}$$

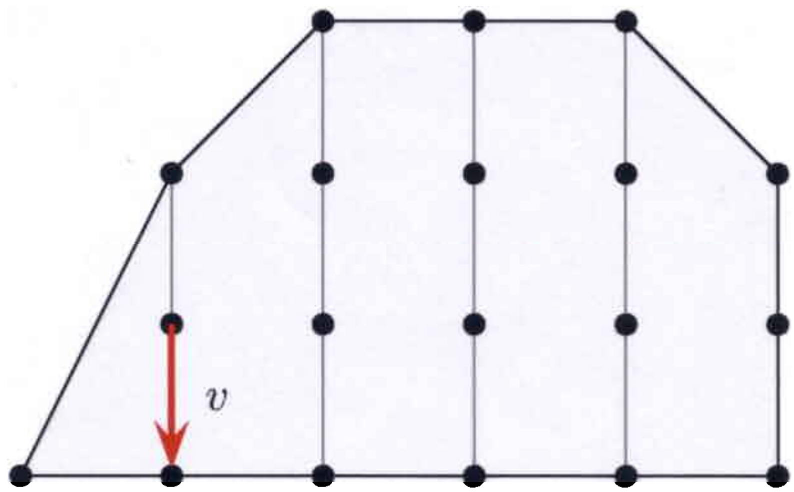
$$0 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 0$$

(i) $[E(R), E(R)] = E(R)$

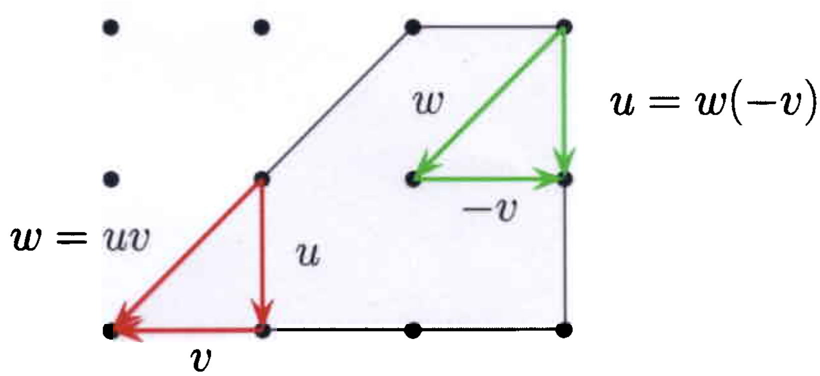
(ii) $Z(E(R)) = 1$

(iii) universal central extension

(iv) $K_2(R) = Z(St(R))$.

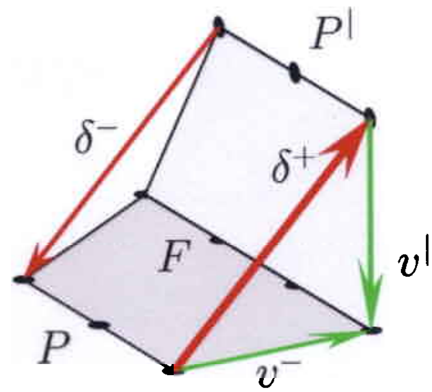


A column structure

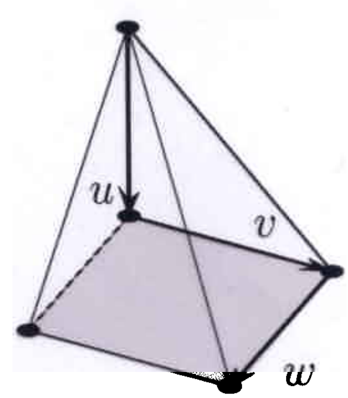


The product of column vectors

A nonbalanced polytope



Doubling along a facet



A difficult polytope

6. Higher K-groups

Quillen's + construction

$$K_i^Q(R,P) = \pi_i\left(B\mathbb{E}(R,P)^+\right) \quad i \geq 2$$

$$= \pi_i\left(B\,\mathbb{S}t(R,P)^+\right) \quad i \geq 3$$

Volodin's construction

requires a suitable notion of "triangular subgroup" = "rigid systems" (in the partial product structure $\mathrm{Co}(P)$)

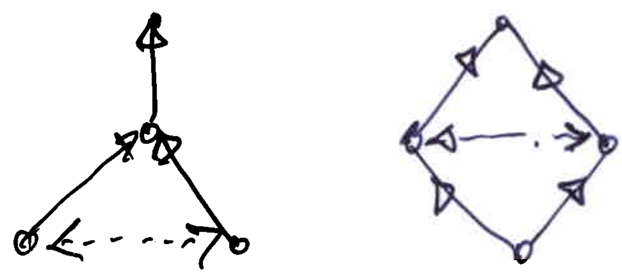
Volodin simplicial set $V(\mathbb{E}(R,P))$

$$K_i^V(R,P) = \pi_{i-1}\left(|V(\mathbb{E}(R,P))|\right) \quad i \geq 2$$

(basepoint = id)

Suslin: Quillen = Volodin for GL_n

Can be generalized to a certain class of polytopes, called <u>col-divisible</u>



<u>Theorem:</u> P col-divisible $\Rightarrow$

$$K_i^Q(R,P) = K_i^V(R,P) \quad \text{for } i \geq 2.$$

<u>Conjecture:</u> P col-divisible

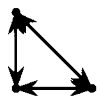
$$\Rightarrow \quad K_i(R,P) = \left(K_i(R)\right)^c$$

$c = \#$ number of equivalence classes of base facets

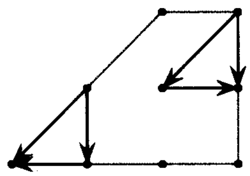
$$F \approx G \Leftrightarrow \text{ex. } v, -v \in Col(P) \text{ s.t.}$$

$$F = P_v, \quad G = P_{-v}$$

The classification of balanced polygons



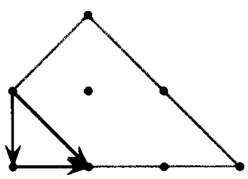
$\{\pm u, \pm v, \pm w\}$ $\quad K_2$



$\{u, \pm v, w\}$

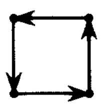
$w = uv$

$u = w(-v)$ $\quad K_2 \oplus K_2$

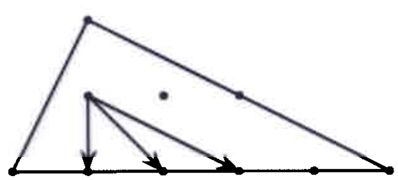


$\{u, v, w\}$

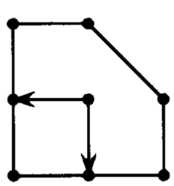
$w = uv$ $\quad K_2 \oplus K_2$



$\{\pm u, \pm v\}$ $\quad K_2 \oplus K_2$



$\{v : P_v = F\}$ $\quad K_2$



$\{u, v\}$ $\quad K_2 \oplus K_2$

Winfried Bruns: From polytopal algebras to polyhedral K-theory

– Joint work with J. Gubeladze

K field

P lattice polytope = (convex hull) $\mathrm{conv}(x_1, \dots, x_m)$, $x_i \in \mathbb{Z}^n$

$K[P]$ polytopal algebra

$K[\text{lattice points in } P] \,/\, $(binomials representing affine dependencies of lattice points in P)

$E_P = \{(x,1) : x \in P \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^{n+1}$

$S_P = \mathbb{Z}_+ E_P \subset \mathbb{Z}^{n+1}$

(see the first transparency)

$K[P] = K[S_P]$.

$P = \Delta_{n-1}$, $K[\Delta_{n-1}] = K[X_1, \dots, X_n] = \mathrm{Sym}(K^n)$

$\mathrm{Vect}(K) \longleftrightarrow \left\{ \begin{matrix} K[X_1, \dots, X_n] \\ \text{graded } K\text{-hom} \end{matrix} \right\} \rightsquigarrow \left\{ \begin{matrix} K[P] \\ \text{graded } K\text{-hom} \end{matrix} \right\}$

1. gr. aut$(K[P])$

~~(see the second transparency)~~

Theorem: Every $\gamma \in$ gr. aut$(K[P])$ has a decomposition
$$\gamma = \varepsilon \circ \pi \circ \delta$$

where δ ~~#~~ toric automorphism

π induced by symmetry of P

ε product of elementary automorphisms

in $GL(n, K)$,

$\delta \Longleftrightarrow$ diagonal matrix

$\pi \Longleftrightarrow$ permutation matrix

$\varepsilon \Longleftrightarrow$ product of elementary matrices

---

$v \in \mathbb{Z}^n$ column vector of $P \Leftrightarrow \exists$ facet F of P s.t. $x + v \in P$ for all $x \in \mathbb{Z}^n \cap P$, $x \notin F$.

(see the ~~we call first a "base facet" of~~ second transparency: column structure, etc.)

$F = P_0$ base facet of v

G facet of P

$ht_G(x)$ = lattice height of x above G

$\langle G, v \rangle = ht_G(x+v) - ht_G(x)$.

Then $v \in \mathrm{Col}(P) \Leftrightarrow \exists$ facet F s.t. $\langle G, v \rangle \geq 0$ for $G \neq F$ and

$\langle F, v \rangle = -1$.

Bruns (3)

R ring, commutative $\quad R[P]$

$\lambda \in R$

$e_v^\lambda(x) := (1+\lambda x)^{ht_F(x)} \cdot x \qquad F = P_v$

elementary automorphism

See the third transparency: Milnor's K_2

2. Milnor's K_2:

3. Polytopal Steinberg relations

$u, v \in Col(P)$ We say uv exists $\Leftrightarrow$ $u+v \in Col(P)$, $P_{uv} = P_u$

Def: P is balanced if $\langle P_u, v\rangle \leq 1$ for all $u, v \in Col(P)$.

See the fourth transparency: a nonbalanced polytope, etc.

Proposition: P balanced

$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists} \\ e_{vu}^{\lambda\mu} & \text{if } vu \text{ exists} \\ 1 & \text{if } u+v \neq 0 \end{cases}$$

4. Doubling Spectra

$P = P^- \qquad P' = rot_{F, 90°}(P^-)$

$P^{\dashv_F} = conv(P^-, P')$

Bruns (4)

Proposition: P balanced

$$\mathrm{Col}(P^{\dashv}) = \mathrm{Col}(P^-) \cup \mathrm{Col}(P') \cup \{\delta^+, \delta^-\}$$

$$F = P_v, \quad \cancel{v} = v^- = \delta^+ v'$$
$$v' = \delta^- v^-$$

doubling spectrum $= (P_i,\ i = 0, \dots, \infty)$

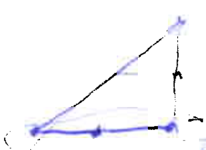
where $P_{i+1} = P_i^{\dashv F_i}$

Each facet is used for doubling infinitely many times.

5. Polytopal K_2:

$$P = \bigcup_i P_i, \quad \mathrm{Col}(P) = \bigcup_i \mathrm{Col}(P_i)$$

$\mathbb{E}(R, P) =$ subgroup of gr. aut. $(R[P])$ generated by all e_v^λ, $v \in \mathrm{Col}(P)$, $\lambda \in R$.



Theorem: $Z(\mathbb{E}(R,P))$ trivial, $\mathbb{E}(R,P)$ perfect group.

$$\mathrm{St}(R,P) = \langle x_v^\lambda,\ v \in \mathrm{Col}(P),\ \lambda \in R \rangle / \text{(Steinberg relations)}$$

Theorem: P balanced

$$0 \to K_2(R,P) \to \mathrm{St}(R,P) \to \mathbb{E}(R,P) \to 0$$ is a universal central extension

Bruns (5

Also, $K_2(R,P) = Z(St(R,P))$.

See the 5th transparency: "Higher K-groups"

6th transparency: Sudim...

~~7th transparency:~~