

MIK1 Feb 6 ①  
Paul Roberts: Almost flat extensions

Faltings: "Almost étale extensions"  
(Astérisque 279)

Graber-Ramero: "Almost ring theory"  
(Arxiv)

J. Tate: "p-divisible groups" (1967)

Heitman: proved the "Direct Summand Conjecture" in  
mixed characteristic, dimension 3.

Almost flatness:

Define a <sup>full</sup> subcategory of the category of  $R$ -modules  
( $R$  commutative), which are called "almost zero".  
Denote this  $\mathcal{C}$ .

(1)  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  short exact,  
then  $M \in \mathcal{C} \Leftrightarrow M' \in \mathcal{C}$  and  $M'' \in \mathcal{C}$  (<sup>Serre</sup> subcategory)

(2) If  $M_\alpha \in \mathcal{C}$ , (~~direct set~~),  $\varinjlim M_\alpha \in \mathcal{C}$ .

Say a module  $M$  is almost flat if  $\text{Tor}_1^R(M, N) \in \mathcal{C}$  for all  $N$ ,  $\forall i >$

i.e.  $-\otimes M$  takes exact sequences to  
sequences with homology in  $\mathcal{C}$ .

More generally, take  $R \rightarrow S$ ,  $\mathcal{C}$  a full subcategory of the cat. of  $S$ -modules,  $M$  an  $S$ -module, require  $\text{Tor}_i^R(M, N) \in \mathcal{C}$  for all  $R$ -modules  $N$ ,  $\forall i \geq 0$ .

Interesting categories  $\mathcal{C}$ :

$R \rightarrow S$   
 Notetherian \ not Notetherian but integral over  $R$ .

1. (Faltings, Heitmann) "powers of  $p$ ":

$R, S$  have mixed characteristic  $p$  ("arithmetic case")  
 $S$  contains small powers of  $p$ : i.e.  $p^n \in S$  for  $n \gg 0$  (intgr.)  
 $M \in \mathcal{C} \Leftrightarrow p^n M = 0$  for  $n \gg 0$ .

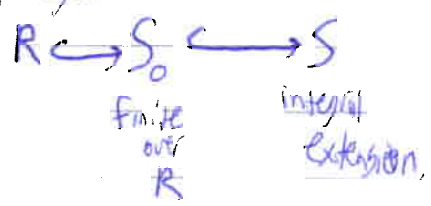
2. (Gabber-Romero): "idempotent ideal"

or ideal of  $S$  with  $\alpha^2 = \alpha$ .  
 $M \in \mathcal{C} \Leftrightarrow \alpha M = 0$ . (This is what Quillen calls "almost nil" modules)

3. "with respect to  $\alpha$  valuation  $v$  on  $S$ " (additive)

$M \in \mathcal{C} \Leftrightarrow \forall m \in M \forall \epsilon > 0, \exists \alpha \in S$  with  $v(\alpha) < \epsilon$  and  $\alpha m = 0$ .

$R$  regular local



If  $S$  is almost flat over  $R$ ,  $\text{Tor}_i^R(R, S) \in \mathcal{C}$  (for any existing mod. ops.)

$x_1, \dots, x_d$  any system of parameters of  $R$ ,  $H_i(K(x_1, \dots, x_d) \otimes_R S) \in \mathcal{C}$ ,  $i \geq 0$ .

Also, Local cohomology  $H_{m_R}^i(S) \in \mathcal{C}$ ,  $i < d$ .  
 (by the direct limit condition)

Call this property "almost Cohen-Macaulay".

If ~~for all~~  $R \rightarrow S_0$  <sup>(regular finite extension)</sup>  $\exists$  an almost CM extension  $S$ ,  
 for any of the ~~three~~ above 3 choices of  $\mathcal{C}$ ,  
 (but in (3) we restrict  $v$  to have the property that  $v(s) > 0 \forall s \in m_{S_0}$ ).  
 Then the "Homological conjectures" hold

- e.g., Direct Summand, Canonical Element

(Heitmann basically showed this for  $e = \text{case (1)}$ , but in fact his proof goes through for case (3), which is weaker.)

(This has been known for a long time if you remove the word "almost"...) 

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Do such extensions exist?

1. Positive characteristic  $p$ :

$$R \rightarrow S_0 \hookrightarrow \bigcup S_0^{1/p^n} = S$$

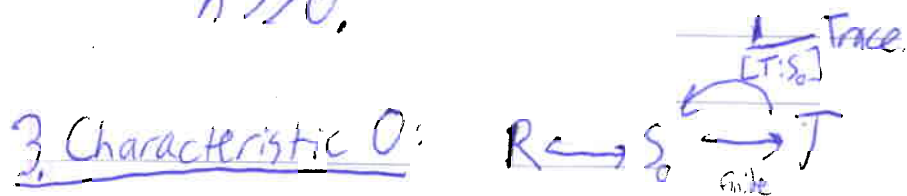
$\exists$  nonzero  $c$  (e.g. a test elt.) such that  $\mathcal{O}_c = \langle c^{1/p} \rangle$   
 makes  $S$  almost Cohen-Macaulay (for  $\mathcal{C} = \text{case (2)}$ )

Theorem (Hochster-Huneke):  $S_0^+ = \text{"complete integral closure"}$   
 $= R^+$  is Cohen-Macaulay

2. Mixed characteristic  $p$

Heitmann proved that if  $\dim(S_0) = 3$ , then  $S_0^+$  is almost Cohen-Macaulay, with respect to small powers of  $p$ .

If  $p, x, y$  is a system of parameters, and if for some  $N \in \mathbb{N}$   $p^N z \in (x, y)$ , then  $p^{1/n} z \in (x, y) S_0^+$  for  $n \gg 0$ .



$S_0^+$  is not Cohen-Macaulay.

Let  $\dim(S_0) = 3$ . (Then  $S_0^+$  could be, conceivably, almost CM in  $e = \dim S_0$ .)  
 Is  $H_{m_x}^2(S_0^+)$  almost zero with respect to a valuation?

Example:  $S_0 = \frac{k[x, y, z, w]}{(x^2 - zw)} \left[ \eta, \frac{w}{x} \eta, \frac{w^2}{x^2} \eta \right]$  is integrally closed,  
 where  $\eta^2 = (x - \alpha_1 z)(x - \alpha_2 z)(x - \alpha_3 z)(x - \alpha_4 z)$ ,  $\alpha_i \in k$  distinct elements.

$x, y, z - w$  is a system of parameters

$$(z - w) \left( \frac{w}{x} \eta \right) = Y \eta - X \left( \frac{w^2}{x^2} \eta \right), \text{ so } S_0 \text{ is not CM.}$$

Use the valuation associated to the grading of  $S_0$ .

$$v(\sqrt{x - \alpha_1 z}) = \frac{1}{2}$$

$$\sqrt{2\sqrt{x+z} - 2\sqrt{x-z}} - \sqrt{\sqrt{x+z} + 2\sqrt{x-z}} \text{ has val. } \frac{1}{8}$$