Paul Roberts: Almost flat extensions

Faltings: "Almost étale extensions"
(Astérisque 279)

Gabber–Ramero: "Almost ring theory"
(Arxiv)

J. Tate: "p-divisible groups" (1967)

Heitmann: proved the "Direct Summand Conjecture" in mixed characteristic, dimension 3.

Almost flatness:

Define a full subcategory of the category of R-modules (R commutative), which are called "almost zero".
Denote this $\mathcal{C}$:

(1) $0 \to M' \to M \to M'' \to 0$ short exact,
then $M \in \mathcal{C} \Leftrightarrow M' \in \mathcal{C}$ and $M'' \in \mathcal{C}$ ("Serre subcatego[ry]")

(2) If $M_\alpha \in \mathcal{C}$, (~~directed set~~), $\varinjlim M_\alpha \in \mathcal{C}$.

Say a module M is almost flat if $\mathrm{Tor}_i(M, N) \in \mathcal{C}$ for all N, $\forall i >$

i.e. $- \otimes M$ takes exact sequences to sequences with homology in $\mathcal{C}$.

More generally take $R \to S$, $\mathcal{C}$ a full subcategory of the cat. of S-modules, M an S-module, require $\mathrm{Tor}_i^R(M,N) \in \mathcal{C}$ for all R-modules N, $\forall i > 0$.

Interesting categories $\mathcal{C}$:

$R \to S$
Noetherian — not Noetherian, but integral over R.

1. (Faltings, Heitmann) "powers of p":

   R, S have mixed characteristic p ("arithmetic case")
   S contains small powers of p: i.e. $p^{1/n} \in S$ for $n \gg 0$ (integer)
   $M \in \mathcal{C} \Leftrightarrow p^{1/n} M = 0$ for $n \gg 0$.

2. (Gabber-Romero): "idempotent ideal"
   $\mathfrak{a}$ ideal of S with $\mathfrak{a}^2 = \mathfrak{a}$.
   $M \in \mathcal{C} \Leftrightarrow \mathfrak{a} M = 0$. (This is what Quillen calls "almost nil" modules)

3. "with respect to a (additive) valuation v on S"
   $M \in \mathcal{C} \Leftrightarrow \forall m \in M\ \forall \varepsilon > 0$, $\exists s \in S$ with $v(s) < \varepsilon$ and $sm = 0$.

$R, \mathfrak{m}, k$ regular local

$R \hookrightarrow S_0 \hookrightarrow S$
S_0 finite over R; S integral extension.

If S is almost flat over R, $\mathrm{Tor}_i^R(k, S) \in \mathcal{C}$ (for any $\mathcal{C}$ satisfying theorems)

$x_1, \ldots, x_d$ any system of parameters of R, $H_i(K.(x_1,\ldots,x_d) \otimes_R S) \in \mathcal{C}$, $i > 0$.

Roberts (3)

A(a), Local cohomology $H^i_{m_R}(S) \in \mathcal{C}$, $i<d$.

(by the direct limit condition)

Call this property "almost Cohen-Macaulay".

IF for all (regular finite extension) $R \hookrightarrow S_0$, $\exists$ an almost CM extension S, for any of the above 3 choices of $\mathcal{C}$, (but in (3) we restrict v to have the property that $v(s)>0$ $\forall s \in m_S$).

Then the "Homological conjectures" hold

– e.g. Direct Summand, Canonical Element

(Heitmann basically showed this for $\mathcal{C}$ = case (1), but in fact his proof goes through for case (3), which is weaker.)

(This has been known for a long time if you remove the word "almost"...)

Do such extensions exist?

1. Positive characteristic p:

$$R \hookrightarrow S_0 \hookrightarrow \bigcup S_0^{1/p^n} = S$$

$\exists$ nonzero c (e.g. a test elt) such that $\mathfrak{a} = \langle c^{1/p^n} \rangle$ makes S almost Cohen-Macaulay (for $\mathcal{C}$ = case (2))

Theorem (Hochster–Huneke): S_0^+ = "complete integral closure" $= R^+$ is Cohen-Macaulay

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Roberts (4)

2. Mixed characteristic p

Heitmann proved that if $\dim(S_0)=3$, then S_0^+ is almost Cohen-Macaulay, with respect to small powers of p.

If p, x, y is a system of parameters, and if for some $N\in\mathbb{N}$ $p^N z\in(x,y)$, then $p^{1/n}z\in(x,y)S_0^+$ for $n>>0$.

3. Characteristic 0:

$$\begin{tikzcd} R \arrow[r, hook] & S_0 \arrow[r, "\text{finite}"'] & T \arrow[l, bend right, "\frac{1}{[T:S_0]}\text{Trace}"'] \end{tikzcd}$$

S_0^+ is not Cohen-Macaulay.

Let $\dim(S_0)=3$. (Then S_0^+ could be, conceivably, almost CM for $e=\omega x 3$.)

Is $H^2_{m_R}(S_0^+)$ almost zero with respect to a valuation?

Example: $S_0=\dfrac{k[X,Y,Z,W]}{(XY-ZW)}\left[\eta, \dfrac{W}{X}\eta, \dfrac{W^2}{X^2}\eta\right]$ is integrally closed, where $\eta^2=(X-\alpha_1Z)(X-\alpha_2Z)(X-\alpha_3Z)(X-\alpha_4Z)$, $\alpha_i\in k$ distinct elements

$X, Y, Z-W$ is a system of parameters

$$(Z-W)\left(\frac{W}{X}\eta\right)=Y\eta-X\left(\frac{W^2}{X^2}\eta\right),$$ so S_0 is not CM.

Use the valuation associated to the grading of S_0.

$$v\left(\sqrt{X-\alpha_1Z}\right)=\tfrac{1}{2}.$$

$$\sqrt{2\sqrt{X+Z}-2\sqrt{X-Z}}-\sqrt{\sqrt{X+Z}+2\sqrt{X-Z}}$$ has val. $\frac{1}{8}$