Jerzy Weyman: Applications of quiver representations: I

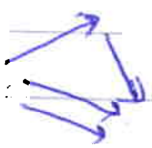
Quiver = oriented graph

$Q = (Q_0, Q_1)$

Q_0 = vertices = $\{x, y, z, \dots\}$

Q_1 = arrows = $\{a, b, c, \dots\}$

$a : \overset{\text{tail}}{ta} \to \overset{\text{head}}{ha}$



$\mathrm{Rep}_K(Q) \ni V = \left\{ \begin{array}{l} V(x);\ x \in Q_0, \quad V(x) \in \mathrm{Vect}_K \ (\text{finite dim'l}) \\ V(a) : V(ta) \to V(ha) \end{array} \right\}$

Def: $\dim V = \alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$ if $\alpha(x) = \dim V(x)$

For $V, W \in \mathrm{Rep}_K(Q)$,

$\mathrm{Hom}_Q(V, W) = \{ f(x) : V(x) \to W(x) \mid x \in Q_0$

such that all diagrams of the form:

$$\begin{tikzcd} V(ta) \arrow[r, "V(a)"] \arrow[d, "f(ta)"'] & V(ha) \arrow[d, "f(ha)"] \\ W(ta) \arrow[r, "V(b)"'] & W(ha) \end{tikzcd}$$

commute $\}$

Ex: 1) $Q : x \xrightarrow{a} y$

$\mathrm{Rep}_K(Q) = \{ V(x) \xrightarrow{V(a)} V(y),$

$$\begin{tikzcd} V(x) \arrow[r] \arrow[d] & V(y) \arrow[d] \\ W(x) \arrow[r] \arrow[ur, phantom, "?"] & W(y) \end{tikzcd}$$

$\}$

2) $x \circlearrowright a$ (Jordan canonical forms)

$V(x) \circlearrowright V(a)$

KQ = path algebra

$p = a_1 \cdots a_s$ $\quad ha_{i+1} = ta_i$

$e_x : x \to x$ empty path $(x \in Q_0)$

$e_x^2 = e_x$

$$pq = \begin{cases} p \circ q & \text{if } hq = tp \\ 0 & \text{otherwise.} \end{cases}$$

$$\left(pe_x = \begin{cases} 0, & tp \neq x \\ p, & tp = x \end{cases} \right)$$

~~Ex~~: $x \xrightarrow{a} y$

KQ is a 3 dim'l algebra: $Ke_x \oplus Ke_y \oplus Ka$.

~~Rep~~ KQ is finite-dimensional algebra $\Leftrightarrow$ Q has no oriented cycles

$\mathrm{Rep}_K Q \approx KQ$

$\mathrm{Ind}_K(Q) = \{$indecomposable modules over $KQ\}$

$\mathrm{Ind}_K(Q, \alpha) = \{$ " " " with dimension vector $\alpha\}$

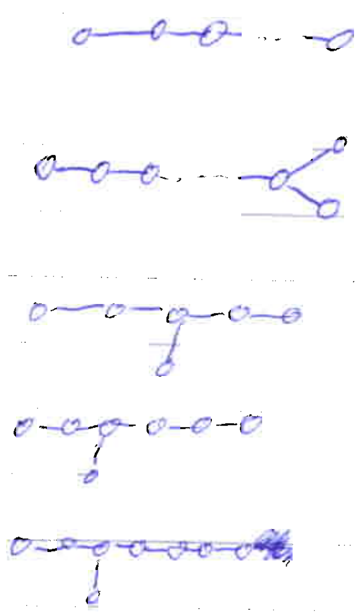
$K^m \xrightarrow{\varphi} K^n$: $\quad r(K \xrightarrow{id} K) \oplus (m-r)(K \to 0) \oplus (n-r)(0 \to K)$

$\Gamma(Q)$ = graph, forgetting orientation

Gabriel:

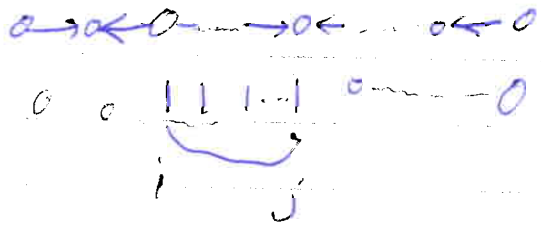
$\mathrm{Ind}_K(Q)$ finite $\Longleftrightarrow$

$\Gamma(Q) = \begin{cases} A_n \\ D_n \\ E_6 \\ E_7 \\ \text{or } E_8 \end{cases}$



$\mathrm{Ind}_K(Q) \longleftrightarrow$ positive roots

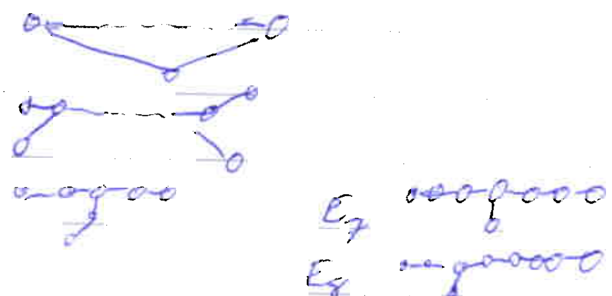
Ex:



E_{ij}

$\mathrm{Ind}_K(Q)$ tame $\Longleftrightarrow$ 1-parameter families in $\mathrm{Ind}_K(Q)$

$\Longleftrightarrow$ A_n, D_n, E_6, E_7, E_8



Kac (1980): Q - has no oriented cycles

$\{\alpha \mid \mathrm{Ind}_K(Q,\alpha) \neq \emptyset\}$ does not depend on orientation of arrows

Roots = dimension vectors of indecomposable modules

KQ - hereditary!

i.e. $V, W \in \mathrm{Rep}_K(Q)$

$$\bigoplus_{x\in Q_0} \mathrm{Hom}_K(V(x),W(x)) \xrightarrow{d_W^V} \bigoplus_{a\in Q_1} \mathrm{Hom}_K(V(ta),W(ha))$$

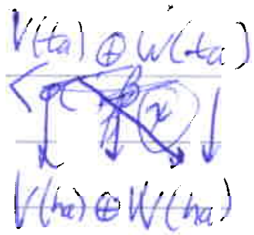
$$d_W^V\{(\varphi(x))\} \longmapsto \{(\varphi(ha)\circ V(a) - W(a)\circ \varphi(ta))\}$$

$$\begin{tikzcd} V(ta) \arrow[r, "V(a)"] \arrow[d, "\varphi(ta)"'] & V(ha) \arrow[d, "\varphi(ha)"] \\ W(ta) \arrow[r, "W(a)"'] & W(ha) \end{tikzcd}$$

We have $\ker d_W^V = \mathrm{Hom}_Q(V,W)$

Also, $\mathrm{coker}\, d_W^V = \mathrm{Ext}_Q(V,W)$

~~Euler form~~:



Euler form: $\langle \alpha,\beta\rangle = \sum_{x\in Q_0} \alpha(x)\beta(x) - \sum_{a\in Q_1} \alpha(ta)\beta(ha)$

nonsymmetric bilinear form

$(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$

Example: $1 \xrightarrow{a} 2 \xrightarrow{b} 3$

$$V = (K \to K \to 0)$$

$$W(1) \oplus W(2) \xrightarrow{\begin{pmatrix} W(a) & -I_d \\ & W(b) \end{pmatrix}} W(2) \oplus W(3)$$

$$W(1) \xrightarrow{(W(b)\,W(a))} W(3)$$

$Rep_K(Q, \alpha)$

$GL_K(Q, \alpha)$

$$\{\text{Iso. classes in } Rep_K(Q)\}$$
$$\updownarrow$$
$$\{GL_K(Q,\alpha)\text{-orbits}\}$$

Q – Dynkin (i.e. A_n, D_n, E_6, E_7, E_8)

$Rep_K(Q, \alpha)$ Finitely many orbits

? Geometry of orbit closures ?

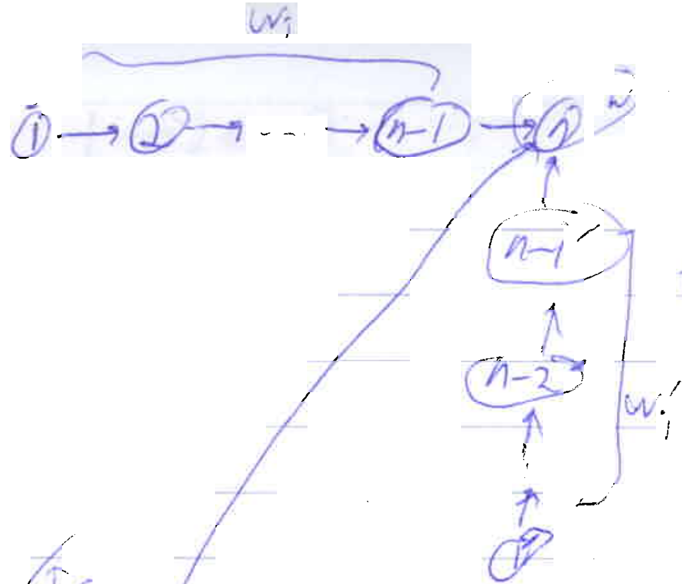
Ex: ~~Ex~~ $K^m \longrightarrow K^n$

orbit closures = determinantal varieties

Ex $1 \to 2 \to \cdots \to n$

$V(1) \to V(2) \to \cdots \to V(n)$

orbits given by ranks of compositions

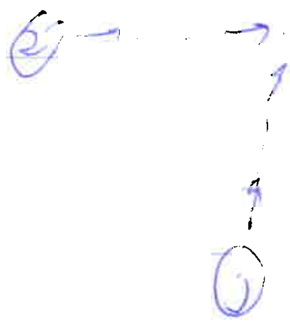


(numbers represent dimension)

(If all arrows are injective, we get two flags)

$G = GL_n$

$G/B \times_G G/B$



$W_i \oplus W_j' \longrightarrow W.$

G. Zwara, G. Bobinski: A_n, D_n: all orbit closures are normal, CM, rat'l sings, and rank conditions give reduced equations

$1 \rightarrow 2 \leftarrow 3$

A-R quiver:

$$\begin{tikzcd}[column sep=small]
 & KKO \arrow[dr] & & OOK \\
OKO \arrow[ur] \arrow[dr] & & KKK \arrow[ur] \arrow[dr] & \\
 & OKK \arrow[ur] & & KOO
\end{tikzcd}$$

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$

A-R quiver:

$$\begin{tikzcd}[column sep=tiny]
 & & & KKKK \arrow[dr] & & & \\
 & & 0KKK \arrow[ur] \arrow[dr] & & KKK0 \arrow[dr] & & \\
 & 00KK \arrow[ur] \arrow[dr] & & 0KK0 \arrow[ur] \arrow[dr] & & KK00 \arrow[dr] & \\
000K \arrow[ur] & & 00K0 \arrow[ur] & & 0K00 \arrow[ur] & & K000
\end{tikzcd}$$

Thm (Zwara) A, B two fin-dim'l algebras $/ K$

$\mathcal{F} : B\text{-Mod} \rightarrow A\text{-mod}$ ~~exact f~~

1) $\mathcal{F}$ exact
2) $\dim_K \operatorname{Hom}_A(\mathcal{F}U, \mathcal{F}V) - \dim_K \operatorname{Hom}_B(U, V) = \overset{\text{bilinear}}{\xi}(\dim U, \dim V)$ for $U, V \in B\text{-Mod}$.

(i.e. $\mathcal{F}$ is "Hom controlled")

$U, V \in B\text{-mod}$

$\overset{\text{orbit}}{O_U} \subset \overline{O}_V, \quad O_{\mathcal{F}U} \subset \overline{O}_{\mathcal{F}V}$

$\operatorname{Sing}(\mathcal{F}U, \mathcal{F}V) = \operatorname{Sing}(U, V)$

$$\begin{tikzcd}[column sep=small]
 & (Z, z) \arrow[dl, "f"'] \arrow[d, "g"] \\
(X, x) & (Y, y)
\end{tikzcd}$$

f, g smooth.

Embeddings $\rightarrow$ reduce to equioriented case (Lakshmibai/Magyar 1998)

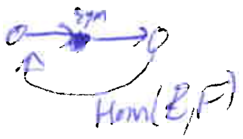
Weyman (8)

$D_n \rightsquigarrow D^{eq}_{2n-1}$. (Brion)

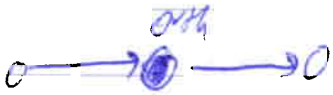
(Can one prove these theorems with comm. alg. and Gröbner bases?)

Symmetric Quivers

symplectic / orthogonal figure for symmetric



orthogonal and symplectic representations



S. Lovett:

Codimension 4 Gorenstein:

all are tame / fail

→ determinantal ($n \times n$ minors of $(n+1) \times (n+1)$ matrix) — $o \to o$

→ cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Segre embedding — $o \rightrightarrows o$

→ Huneke-Ulrich deviation 2 ideals — $o \to o \xrightarrow{+} o$

X - skew symm of $(2n+1) \times (2n+1)$

Y $1 \times (2n+1)$

$$J = I_1(YX) + Pf_{2n}(X)$$

→ F sympl. space

$F \xrightarrow{f} F$ sympl. map

$\dim \operatorname{Ker} f \geq 2$,

$\dim \operatorname{Ker} f^2 \geq 4$.

(Kustin-Miller: ?)