Harm Derksen: Applications of quiver representations II

$Q = (Q_0, Q_1)$ quiver $\qquad K = \overline{K}$ field

Q_0 vertices, Q_1 arrows

$a \in Q_1$, $\quad ta \in Q_0$ "tail"
$\qquad\qquad ha \in Q_0$ "head"

$\mathbb{N}^{Q_0}$ dimension vectors

$\alpha \in \mathbb{N}^{Q_0}$ $\qquad \mathrm{Rep}(Q,\alpha) = \prod_{a \in Q_1} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$

acts

$GL(\alpha) = \prod_{x \in Q_0} GL(\alpha(x))$

$$\left\{ \begin{array}{c} GL(\alpha)\text{-orbits} \\ \text{in } \mathrm{Rep}(Q,\alpha) \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \alpha\text{-dim'l reps} \\ \text{of } Q \end{array} \right\}$$

$I(Q,\alpha) = K[\mathrm{Rep}(Q,\alpha)]^{GL(\alpha)}$ ring of invariants

Theorem (Le Bruyn–Procesi, char 0): $I(Q,\alpha)$ is generated by traces of endomorphisms corresponding to closed paths

(Donkin, char > 0): use coefficients of characteristic polynomial instea[d] char $K = 0$

<u>Example</u>: $Q = \circ \rightleftarrows \circ$

$\alpha = (2, 3)$

$$K^2 \underset{B}{\overset{A}{\rightleftarrows}} K^3$$

$I(Q,\alpha)$ is generated by $Tr(AB), Tr(BA), Tr(ABAB), Tr(BABA), \ldots$

If Q has no oriented cycles, then $I(Q,\alpha) = K$.

<u>Def</u> $SL(\alpha) \subseteq GL(\alpha)$

$$SL(\alpha) = \prod_{x \in Q_0} SL(\alpha(x))$$

$$SI(Q,\alpha) = K[Rep(Q,\alpha)]^{SL(\alpha)}$$ "ring of semi-invariants"

$\sigma \in \mathbb{Z}^{Q_0}$, this defines a multiplicative character of $GL(\alpha)$.
"weights"

$$\sigma : GL(\alpha) \ni \{A(x)\}_{x \in Q_0} \longmapsto \prod_{x \in Q_0} \det(A(x))^{\sigma(x)}$$

$$SI(Q,\alpha)_\sigma = \{ f \in K[Rep(Q,\alpha)] \mid Af = \sigma(A) f \quad \forall A \in GL(\alpha) \}$$

$$SI(Q,\alpha) = \bigoplus_\sigma SI(Q,\alpha)_\sigma,$$ (giving us a multigrading)

<u>Def</u> $\alpha \in \mathbb{N}^{Q_0}$ dim. vector, $\sigma \in \mathbb{Z}^{Q_0}$ weight

$$\sigma(\alpha) = \sum_{x \in Q_0} \sigma(x) \alpha(x)$$

<u>Fact</u> If $SI(Q,\alpha)_\sigma \neq 0$, then $\sigma(\alpha) = 0$.

Example: $Q = \begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r] & \circ & \circ \arrow[l] \\ & \circ \arrow[u] & \end{tikzcd}$, $\alpha = \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array}$

$$\begin{tikzcd} & K^1 \arrow[d, "A_1"] & \\ K^1 \arrow[r, "A_2"] & K^2 & K^1 \arrow[l, "A_4"'] \\ & K^1 \arrow[u, "A_3"] & \end{tikzcd}$$

~~weights~~

$I(Q,\alpha) = K$, but: $SI(Q,\alpha) = K[\det(A_1|A_2), \det(A_1|A_3), \det(A_1|A_4), \det(A_2|A_3), \det(A_2|A_4), \det(A_3|A_4), \det\begin{pmatrix} A_1 & 0 & A_3 & A_4 \\ 0 & A_2 & A_3 & \lambda A_4 \end{pmatrix}\ \forall \lambda \in K$

$\{V(a)\}_{a \in Q_1} = V \in Rep(Q,\alpha)$ (actually a rep w/ a preferred basis in every vec. space)

$\{W(a)\}_{a \in Q_1} = W \in Rep(Q,\beta)$

Exact:
$$0 \to Hom_Q(V,W) \to \prod Hom(K^{\alpha(x)}, K^{\beta(x)}) \xrightarrow{d_W^V} \prod Hom(K^{\alpha(ta)}, K^{\beta(ha)}) \to Ext_Q(V,W) \to$$
$$\{\varphi(x)\} \longmapsto \{W(a)\varphi(ta) - \varphi(ha)V(a)\}$$

Euler form $\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$

Schofield semi-invariants

If $\langle\alpha,\beta\rangle = 0$, $\Rightarrow$ d^V_W is square
$\Rightarrow$ can define $c(V,W) = \det d^V_W$

Fix V. $c^V = c(V,\bullet) \in SI(Q,\beta)_{\langle\alpha,\bullet\rangle}$

Fix W. $c_W = c(\bullet,W) \in SI(Q,\alpha)_{-\langle\bullet,\beta\rangle}$

Thm (—, Weyman)
(Schofield and van den Bergh in a different way)

(Assume Q has no oriented cycles [although this isn't really necessary].)
$SI(Q,\beta)$ is spanned by c^V's (where $V \in Rep(Q,\alpha)$ and $\langle\alpha,\beta\rangle = 0$)

(In the example from these notes page 3, he now gave information on weights, but I couldn't read it.)

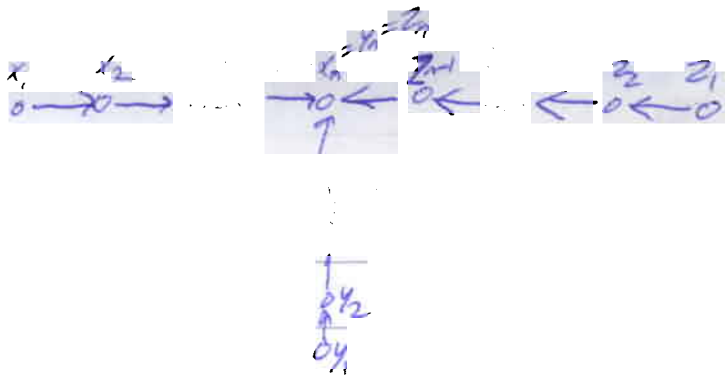
(def) $\Sigma(Q,\alpha) = \{\sigma \in \mathbb{Z}^{Q_0} \mid SI(Q,\alpha)_\sigma \neq 0\}$

Thm (—, Weyman): $\Sigma(Q,\alpha) = \{\sigma \in \mathbb{Z}^{Q_0} \mid \sigma(\alpha) = 0$ and $\sigma(\beta) \leq 0$ $\forall \beta$ with $ext(\beta, \alpha-\beta) = 0\}$

(Def) $ext(\alpha,\beta) = \dim Ext(V,W)$, where $V \in Rep(Q,\alpha)$, $W \in Rep(Q,\beta)$ are in general position.

$\Sigma(Q,\alpha)$ polyhedral cone

Derksen (5)

Example: 

$\alpha =$ 1 2 3 ⋯ $n-1$ n $n-1$ ⋯ 2 1

$n-1$
⋮
2
1

Then $SI(Q,\alpha)_\sigma = (V_\lambda \otimes V_\mu \otimes V_\nu)^{SL(n)}$

← irred reps. of $SL(n)$

Dual partition
$$\lambda' = \left((n-1)^{\sigma(x_{n-1})}, \dots, 1^{\sigma(x_1)}\right)$$
$$\mu' = \left((n-1)^{\sigma(y_{n-1})}, \dots, 1^{\sigma(y_1)}\right)$$
$$\nu' = \left((n-1)^{\sigma(z_{n-1})}, \dots, 1^{\sigma(z_1)}\right)$$

(Littlewood-Richardson coeffs)

$$\Sigma(Q,\alpha) \simeq \{(\lambda,\mu,\nu) \mid c^{\nu}_{\lambda,\mu} \neq 0\}$$

$c^{\nu}_{\lambda,\mu}$ = mult: $V_\nu \subseteq V_\lambda \otimes V_\mu$.

Cor: If $N > 1$, $c^{N\nu}_{N\lambda, N\mu} \neq 0 \Rightarrow c^{\nu}_{\lambda,\mu} \neq 0$.

Knutson–Tao: This is a step to prove Horn's conjecture (note) $c^V(W)=0 \iff c^W(V)=0 \iff \mathrm{Hom}_Q(V,W)\neq 0$

$\iff \mathrm{Ext}_Q(V,W)\neq 0.$

(Def.) α is a <u>Schur root</u> if a general <u>representation of</u> dim α is <u>indecomposable</u>.

<u>Example</u>: Q: one vertex with a loop, $\alpha=(n)$, $n>1$

general matrix is diagonalizable

$\Rightarrow$ (n) is not a Schur root.

$J=\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ Jordan block is indecomposable.

$\Rightarrow$ (n) is a root.

Def: $\alpha \perp\!\!\!\perp \beta$ ("<u>strongly orthogonal</u>") if

$\dim SI(Q, q\beta)_{\langle p\alpha, \cdot\rangle} = \dim SI(Q, p\beta)_{\langle \cdot, q\beta\rangle} = 1$, $\forall p, q \ldots$

$\alpha_1, \alpha_2, \ldots, \alpha_r$ is a <u>Schur sequence</u> if

(1) $\alpha_1, \ldots, \alpha_r$ are Schur roots, and

(2) $\alpha_i \perp\!\!\!\perp \alpha_j$, $\forall i<j$.

If $\alpha=\sum r_i\alpha_i$, $\alpha_1, \ldots, \alpha_r$ Schur sequence,

then $\{\sigma \in \mathbb{Z}^{Q_0} \mid \sigma(\alpha_1)=\cdots=\sigma(\alpha_r)=0\}$ is a face of $\mathcal{E}(Q,\alpha)$

and all faces arise in this way.

$p(n) = \dim SI(Q,\sigma)_{n\sigma}$

Thm (—, Weyman): $p(n)$ is a polynomial.

$R = \bigoplus_{n \geq 0} SI(Q,\sigma)_{n\sigma}$

$\cancel{H(\bigoplus_{n\geq 0} SI(Q,\sigma)_{n\sigma})}\ H(R,t) = \sum_n p(n)t^n = \dfrac{a(t)}{b(t)}$

For the Thm. to be true, we need $b(t) = (1-t)^r$ and $\deg a(t) < \deg b(t)$.

It's not obvious why these hold.

Generalized Fulton

Conj: If $p(1)=1$ then $p(n)=1$ th

Buch Conj: Does $p(n)$ have nonneg. coefficients?