MSRI, Feb 6

Idun Reiten: Hereditary categories over arbitrary fields

I. Classification theorem (w/ M. v.d. Bergh, J.AMS 2002)

(1) Examples and Setup

Setup: $\mathcal{H}$ hereditary abelian (i.e. $\mathrm{Ext}^2_{\mathcal{H}}(-,-)=0$) — Yoneda Ext

k-category ($k=\bar{k}$), $\mathrm{Hom}(A,B)$, $\mathrm{Ext}^1(A,B)$ fin. dim. over …

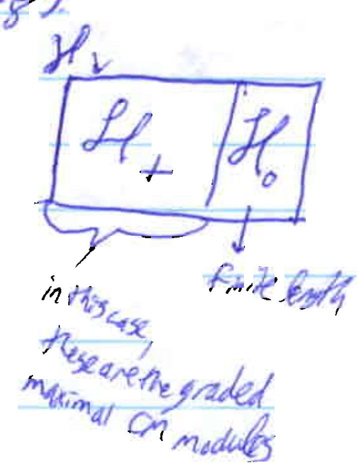
Ex: $\mathrm{mod}\,H$ (f.g. modules) (H fin. dim. her. k-algebra); i.e. $\mathrm{rep}_k Q$, Q quiver (no oriented cycles)

$\mathrm{mod}\,kQ$

Ex: $\mathrm{coh}\,\mathbb{P}^1(k) \sim \mathrm{gr.mod}\,k[X,Y] \big/ \text{fin. length} = \mathrm{qgr}(k[X,Y])$

Ex $R = k[X,Y,Z]/(X^2+Y^3+Z^5)$ (E_8)

$\mathrm{qgr}(R)$



(2) Serre duality and almost split seq.

$\mathrm{mod}\,H$ has almost split sequences (indeed true for any fin dim a…

$[0 \to A \to B \to C \to 0$ a.s.s. if A, C indec. and any

$A = \tau C$

non-isomorphism to C lifts to a map to B.

$\rightsquigarrow$ AR quiver

$$\begin{tikzcd} & B_1 \arrow[dr] \arrow[d, dash, dotted] & \\ A \arrow[ur] \arrow[dr] & {} & C \\ & B_n \arrow[ur] & \end{tikzcd}$$

$A = \tau C$ (indec)

• $\mathcal{H} = \mathrm{coh}\, \mathbb{P}^1(k)$ has Serre duality

(i.e.: $\exists\, \tau : \mathcal{H} \to \mathcal{H}$ s.t. $\mathrm{Ext}^1(A, \tau B) \simeq D(\mathrm{Hom}(B,A))$) (vector space dual)

(note: Serre duality $\Rightarrow \exists$ of a.s.s.)

Def (BK) $\mathcal{H}$ has Serre duality if $\exists$ eq. $\tau : D^b(\mathcal{H}) \to D^b(\mathcal{H})$

Have: $\mathcal{H}$ has Serre d. $\Leftrightarrow$ $\mathcal{H}$ has a.s.s. + (a little more).

Assume: $\mathcal{H}$ has Serre d.

3 Classification theorem

$\mathcal{H}$ with Serre d., noeth. case.

No proj

(i) $\mathrm{coh}\, X \subset \{ q.gr(R) \mid R = k + R_1 + \cdots + R_n + \cdots$ ($\neq 0$), 2-dim'l CM comm. domain, isolated singularity $\}$.

(nonsing. proj. curve)

[$\mathrm{coh}\, \mathbb{P}^1(k) \subset \{ R$ simple hypersurf. sing. A_n, D_n, E_i $(i = 6,7,8) \}$]

(ii) $q.gr_{\mathbb{Z}\times\mathbb{Z}}\, k[X,Y]$ (+ twisted version)

(iii) $\boxed{\mathcal{H} = \mathcal{H}_0}$ (easy)

$\exists$ proj

(i) rep Q (= f.d. rep) (Q finite quiver, no or. cycles) $\subset$ {Q ... all p... ar... fin...

(ii) Not enough project...

(constructed from Qu...

II. Charact. of $\mathcal{H}$ related to quivers (species) (with Lenzing)

$\mathcal{H}$ with Serre duality, no proj., $\mathcal{H} \neq \mathcal{H}_0$, k arb. field.

(1) Degree/Slope

• rank: $\mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0$ (s.s.), $\mathrm{rk}\, X = \ell_{\tilde{\mathcal{H}}}(\tilde{X})$

• degree: Fix $L \in \mathcal{H}_+$, $\mathrm{rk}\, L = 1$. Define $\deg : K_0(\mathcal{H}) \to \frac{1}{2}$

with properties (i) $\deg L = 0$

(ii) $\deg X \geq \ell(X)$ for $X \in \mathcal{H}_0$.

(others)

• $\mathcal{H}$ is domestic if $\delta_{\mathcal{H}} \overset{\mathrm{def}}{=} \deg(\tau L) < 0$.

$0 \to \tau L \to E \to L \to 0$ a.s.s.

• slope (in $\mathcal{H}_+$) $\boxed{\mu = \deg / \mathrm{rk}}$

• $X \neq 0$ in $\mathcal{H}_+$ is stable if $Y \subsetneq X$, then $\mu(Y) < \mu(X)$

Property: $\mu(\tau X) = \mu(X) + \delta_{\mathcal{H}}$.

(2) Charact. of domestic $\mathcal{H}$

Theorem: The foll. are equiv. for $\mathcal{H}$.

(a) $\mathcal{H}$ is domestic

(b) Each ind. X in $\mathcal{H}_+$ is stable

(c) " " " " " " is exceptional (i.e. $\mathrm{Ext}^1(X,X) = 0$)

(d) $\nexists$ oriented cycles ($X_1 \ldots X_n$ in $\mathcal{H}_+$. Note, no 0, no iso.)

(e) $\exists$ a component of bdd. rank in AR-quiver for $\mathcal{H}_+$.

(f) Rank is bounded on $\mathcal{H}_+$

Illustrate part of proof: Assume (a) and (b), show (c).
Let X indec. in $\mathcal{H}_+$.

Have $\mathrm{Ext}^1(X,X) \cong D\,\mathrm{Hom}(X, \tau X)$, by Serre duality.

Assume $f \neq 0 : X \to \tau X$
Then $\mu(X) < \mu(\tau X)$.

$\mu(\tau X) = \mu(X) + \delta_{\mathcal{H}} < \mu(X)$, contradiction.
So X is exceptional.

Example: $\mathcal{H} = \mathrm{Coh}\ \mathbb{P}^1(k)$. $\quad 0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0$

$\mathcal{H}_+$: $\cdots \quad \mathcal{O}(-2) \quad \mathcal{O}(-1) \quad \mathcal{O}$ $\qquad$ ∞ number of components

$\mathcal{H}_0$

Rem: R finite C.M. type $\Rightarrow$ Bounded rank on $\mathcal{H}_+ \sim \mathrm{MCM}(R)$
$\Rightarrow$ domestic.

Ex: $R = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)$ $\quad$ [2 indec.] $\qquad R \xrightarrow{\mathbb{H} \otimes_{\mathbb{R}}} \mathbb{H}$

③ rep Q $\quad \mathcal{H}_Q$

$Q: \cdot \rightrightarrows \cdot$

$\mathcal{H}_Q = \mathrm{Coh}\ \mathbb{P}^1(k)$.