MSRI Feb 6

Mitsuyasu Hashimoto: Good modules over reductive groups and commutative algebra

$k = \bar{k}$

G: reductive group $/k$

$T \subset G$ max. torus of G

Δ: a base of the set of roots of G.

B: the negative Borel group

X^+: the set of dominant weights

For $\lambda \in X^+$, the corresponding (1-dim'l) B-module is denoted by $k\lambda$

$\mathbb{L}_\lambda = G \times^B k\lambda \longrightarrow G/B$

is a G-equivariant line bundle, and the set of sections $\nabla_G(\lambda)$ is a G-module. $\Delta_G(\lambda) = \nabla_G(-w_0\lambda)^*$ ← k-dual

w_0: the longest elt. of $W(G)$

$G = GL_n$: $T = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$

$X(T) = \mathrm{Hom}_{\text{alg-g.p}}(T, \underset{\underset{k^\times}{\|}}{GL_1}) = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \longmapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n} \;\middle|\; \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \right\} \cong \mathbb{Z}^n$

$\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n,\ i \neq j \}$,

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$

$\Delta = \{ \varepsilon_i - \varepsilon_{i+1} \}$

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$$B = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}$$

$$X^+ = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \}$$

If $\lambda = (\lambda_1, \ldots, \lambda_n)$, then $\nabla_G(\lambda) \cong L_{\tilde{\lambda}} V$,

$\Delta_G(\lambda) \cong K_\lambda V$,

where $V = k^n$

$\lambda \geq \mu \Leftrightarrow \lambda - \mu$ is a sum of positive roots

<u>Thm</u> (Donkin, Friedlander, Cline-Parshall-Scott)
For a G-module V, TFAE:

1) $\forall \lambda \in X^+$, $\mathrm{Ext}^1_G(\Delta_G(\lambda), V) = 0$
2) $\forall \lambda \in X^+$, $\forall i > 0$, $\mathrm{Ext}^i_G(\Delta_G(\lambda), V) = 0$.
3) For any order-preserving bijective map $f : X^+ \to \mathbb{N}$, $\exists$ filtration $0 = V_0 \subset V_1 \subset \cdots \subset V$ such that $V = \varinjlim V_i$ and $\forall i$, $V_i / V_{i-1} \cong \bigoplus \nabla_G(f^{-1}(i))$

<u>Def.</u> A G-module V is said to be <u>good</u> if 1)–3) are satisfied.

$$\mathcal{A}_G = \{\text{finite dim'l } G\text{-modules}\}$$

$$\mathcal{Y}_G = \{ V \in \mathcal{A}_G \mid V \text{ is good} \}$$

$$\mathcal{X}_G = \{ V \in \mathcal{A}_G \mid V^* \text{ is good} \}$$

$$\omega_G = \mathcal{Y}_G \cap \mathcal{X}_G.$$

Def: Let $\mathcal{A}$ Abelian cat. and $\mathcal{X}, \mathcal{Y}, \omega \subset \mathcal{A}$.

We say that $(\mathcal{X}, \mathcal{Y}, \omega)$ is a weak Auslander-Buchweitz Context (weak ABC for short) if

AB1) $\mathcal{X}$ is closed under ext, epiker, and direct summands in $\mathcal{A}$

AB2) $\mathcal{Y}$ is closed under ext., mono coker, and direct summands in $\mathcal{A}$, ~~and~~

$\mathcal{Y} \subset \hat{\mathcal{X}} = \{ V \in \mathcal{A} \mid \exists\ 0 \to X_n \to \cdots \to X_0 \to V \to 0$ exact, and each $X_i \in \mathcal{X} \}$

AB3) $\omega = \mathcal{X} \cap \mathcal{Y}$, and $\forall X \in \mathcal{X}$, $\exists\ 0 \to X \to W \to X' \to 0$ exact, $W \in \omega$

and $\mathrm{Ext}^i(X, W) = 0$ $\forall W \in \omega$ $\forall X \in \mathcal{X}$, $\forall i > 0$.

Def. If $\hat{\mathcal{X}} = \mathcal{A}$ moreover, then we say that $(\mathcal{X}, \mathcal{Y}, \omega)$ is an ABC

Thm (Ringel) $(\mathcal{X}_G, \mathcal{Y}_G, \omega_G)$ is an ABC of $\mathcal{A}_G$

Let $(A, \mathfrak{m})$ be a CM local ring with a canonical module K_A.

Let $\mathcal{A}_A = \{$finite A-modules$\}$

$\mathcal{X}_A = \{$maximal CM A-modules$\}$

$\mathcal{Y}_A = \{ M \in \mathcal{A}_A \mid \mathrm{inj.dim}_A M < \infty \}$

$\omega_A = \mathrm{add}\, K_A$.

$\Rightarrow$ (Auslander-Buchweitz C-M Approx):

$(\mathcal{X}_A, \mathcal{Y}_A, \omega_A)$ is an ABC in $\mathcal{A}_A$.

Thm (Auslander-Buchweitz, Auslander-Reiten)
let $\mathcal{X}, \mathcal{Y}, \omega$ be a weak ABC of $\mathcal{A}$.
Then
1) $\mathcal{Y} = \hat{\omega} = \{V \in \mathcal{A} \mid 0 \to \underset{\in \omega}{W_n} \to \cdots \to \underset{\in \omega}{W_0} \to V \to 0 \text{ exact}\}$.
2) For $M \in \hat{\mathcal{X}}$, $\exists\ 0 \to \underset{\hat{\mathcal{Y}}}{Y} \to \underset{\mathcal{X}}{X} \to M \to 0$ exact
(by the Ext' condition)
3) For $M \in \hat{\mathcal{X}}$, $\exists\ 0 \to M \to \underset{\mathcal{Y}}{Y'} \to \underset{\mathcal{X}}{X'} \to 0$.
4) If $Y \in \mathcal{Y}$, then $\omega\text{-resol.dim}\, Y = \mathcal{X}\text{-resol.dim}\, Y$
$= \omega\text{-proj.dim}\, Y$
$= \mathcal{Y}\text{-proj.dim}\, Y$.

Let Γ be a poset ideal of X^+ which is also a subsemigroup of X^+.

$G = GL_n$, $X^+ = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \cdots \geq \lambda_n\} \supset \Gamma = \{\text{partitions}\}$

$G\text{-Mod}_\Gamma = \{V \in G\text{-Mod} \mid \text{Any simple subquotient of } V \cong \operatorname{Soc}(\nabla_G(\lambda)) \text{ for some } \lambda \in \Gamma\}$.

This is a full subcat of G-mod, closed under ext, subquotients, and $\varinjlim$, and $\otimes$.

$\Rightarrow$ $i: G\text{-Mod}_\Gamma \hookrightarrow G\text{-Mod}$ has a right adjoint Φ_Γ, and

$i\,\Phi_\Gamma k[G] \xrightarrow{\varepsilon} k[G]$ is injective, and

$\operatorname{Im} \varepsilon \subset k[G]$ is a k-subbialgebra of $k[G]$

Let $H = \operatorname{Spec} \operatorname{Im} \varepsilon$.

$\Rightarrow$ $G\text{-Mod}_\Gamma \cong H\text{-Mod}$

(In the Example $G = GL_n$ on page 4, $H = \mathrm{Mat}_n$)

Let V be a f. dim'l H-module, $S = \operatorname{Sym} V$, and assume that S is good
($\Leftrightarrow$ $\operatorname{Sym}_i V$ is good $\forall i$)

$\mathcal{A}^\Gamma_{G,S} = \{S\text{-finite } (H,S)\text{-modules}\}$.

$\mathcal{X}^\Gamma_{G,S} = \{M \in \mathcal{A}^\Gamma_{G,S} \mid M$ is S-projective and $\operatorname{Hom}_S(M,S)$ is a good G-module$\}$

$\mathcal{Y}^\Gamma_{G,S} = \{M \in \mathcal{A}^\Gamma_{G,S} \mid M$ is good as a G-module$\}$

$\omega^\Gamma_{G,S} = \mathcal{X}^\Gamma_{G,S} \cap \mathcal{Y}^\Gamma_{G,S}$

$\Rightarrow$ **Thm** ① $(\mathcal{X}^\Gamma_{G,S}, \mathcal{Y}^\Gamma_{G,S}, \omega^\Gamma_{G,S})$ is an ABC in $\mathcal{A}^\Gamma_{G,S}$.

② $M \in \omega^\Gamma_{G,S} \Rightarrow \exists T \in \omega_G \cap H\text{-Mod}\,\Gamma$ such that $M \cong S \otimes_k T$

Cor If $M \in \mathcal{A}^\Gamma_{G,S}$ and M is perfect of codimension h, and both M and $\mathrm{Ext}^h_S(M,S)$ are good, then $\omega^\Gamma_{G,S}$-resol. dim$(M) = h$.

Example: $S = \mathrm{Sym}(V \otimes W)$, $V = k^m$, $W = k^n$

$G = GL(V) \times GL(W)$
$\quad\quad\cap$
$H = \mathrm{End}(V) \times \mathrm{End}(W)$

$1 \leq t \leq \min(m,n)$

$M = S/I_t$, where

$I_t = (t\text{-minors of } (v_i \otimes w_j))$,

where $v_1, \ldots, v_m$ is a basis of V

and $w_1, \ldots, w_n$ is a basis of W

$\Rightarrow$ (Akin–Buchsbaum–Weyman): S and $\overset{M}{S/I_t}$ are good.

(Hochster–Eagon): M is perfect of codim h, where $h = (m-t+1)(n-t+1)$.

$\mathrm{Ext}^h_S(M,S)$ is good, due to G. Kempf's construction.

$\omega_G \cap H\text{-Mod} = \mathrm{add}\{(\wedge V \otimes \wedge W)^{\otimes n} \mid n \geq 0\}$.

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$\Rightarrow S/I_t$ has an $\omega^{\Gamma}_{G,S}$-resol. of length h.

$m=t \Rightarrow$ Buchsbaum-Rim resol. is such a resol.

Sym V : Good

Example: Let $Q=(Q_0, Q_1)$ be a quiver
$Q_0=\{1,\dots,n\}$, $\alpha=(\alpha_1,\dots,\alpha_n)\in \mathbb{Z}_{\geq 0}^{Q_0}$
$V_i=k^{\alpha_i}$, let $G_i = GL(V_i)$, $SL(V_i)$, Sp_{d_i} (d_i even), SO_{d_i}, or trivial
$G=G_1\times\cdots\times G_n \curvearrowright Rep_k(Q,\alpha)=\prod_{a\in Q_1} Hom(V_{t(a)}, V_{h(a)})$

Then using Akin-Buchsbaum-Weyman filtration, and results of Anderson-Jantzen:
If $char(k)\neq 2$, or $\forall i\ G_i \neq SO_{d_i}$, then Sym V^* is good.

Rem ① If $S=$ Sym V is good $\Rightarrow S^G$ is strongly F-regular (type)
(e.g. $SI(Q,\alpha)$ is F-reg)

② " " " " " $\Rightarrow \forall M \in A^{X^+}_{G,S}$, $H^i(G,M)$ is S^G-finite for all $i\geq 0$.

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Let $S = \mathrm{Sym}\, V$ good, $I \subset S$ a graded perfect G-ideal of codim h,

$A = S/I$, Assume A and $K_A = \mathrm{Ext}_S^h(A, \underbrace{S \otimes \Lambda^{\mathrm{top}} V}_{= K_S})$ are good.

$\mathcal{A}_{G,A} = \{A\text{-finite graded } (G,A)\text{-modules}\}$

$\mathcal{X}_{G,A} = \{M \in \mathcal{A}_{G,A} \mid M$ is a MCM A-module, and $\mathrm{Hom}_A(M, K_A)$ is good$\}$.

$\mathcal{Y}_{G,A} = \{N \in \mathcal{A}_{G,A} \mid \mathrm{inj\,dim}_A N < \infty$, and $\mathrm{Hom}_A(K_A, N)$ is good$\}$

$\omega_{G,A} = \mathcal{X}_{G,A} \cap \mathcal{Y}_{G,A}$.

__Thm__: $(\mathcal{X}_{G,A}, \mathcal{Y}_{G,A}, \omega_{G,A})$ is an ABC of $\mathcal{A}_{G,A}$,

and $M \in \omega_{G,A} \iff \exists T \in \omega_G$ s.t. $M \cong K_A \otimes_k T$

example of a weak ABC which is not an ABC: Let A be a CM local ring.

$\mathcal{X} = \{M \mid G\text{-}\dim_A M = 0\}$

$\mathcal{Y} = \{M \mid \mathrm{pd}_A M < \infty\}$

$\omega = \mathrm{add}\, A$.

This is always a weak ABC, but it is an ABC $\iff$ A is Gorenstein.