MSRI, Feb 7

Hans-Bjørn Foxby: Hyper-homological algebra in commutative algebra

Hartshorne: Residues & Duality

R commutative ring, M, N R-mods

$$\mathrm{Ext}^n(M,N) = H^n(\mathrm{Hom}(P,N)) \quad \text{where } P \xrightarrow{\simeq} M \text{ proj res}$$

$$\cong H^n(\mathrm{Hom}(P,I))$$

$$\cong H^n(\mathrm{Hom}(M,I)) \quad \text{where } N \xrightarrow{\simeq} \text{ inj res}$$

Notation: R-complex $X = \cdots \to X_{\ell+1} \to X_\ell \to X_{\ell-1} \to \cdots$

X, Y R-complexes

$$\mathrm{Hom}(X,Y)_n = \prod_p \mathrm{Hom}(X_p, Y_{n+p})$$

$$\partial^{\mathrm{Hom}(X,Y)} = \mathrm{Hom}(X, \partial^Y) - \mathrm{Hom}(\partial^X, Y)$$

what Lipman and Greenlees call Hom

$$\mathbb{R}\mathrm{Hom}(X,Y) = \mathrm{Hom}(P,Y), \text{ where } P \xrightarrow{\simeq} X \text{ semiproj. res.}$$

$$\mathbb{R}\mathrm{Hom}(X,Y) = \mathrm{Hom}(X,I) \xrightarrow{\simeq} \mathrm{Hom}(P,I) \xleftarrow{\simeq} \mathrm{Hom}(P,Y)$$

$Y \xrightarrow{\simeq} I$ semiinj.

($X \xrightarrow{\alpha} Y$ quasi iso $\overset{\text{def}}{\Longleftrightarrow}$ $H(\alpha): H(X) \xrightarrow{\cong} H(Y)$)

(P semiproj $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Hom}(P, -)$ preserves surj. quasi-iso

$\mathbb{R}\mathrm{Hom}(X,Y)$ is uniquely defined up to ~~quasi iso~~ unique iso in $\mathcal{D}(R)$.

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$X \xrightarrow{\simeq} Y \Rightarrow X \xrightarrow{\simeq} Y$ iso in $\mathcal{D}(R)$, derived cat.

M, N R-mods:

$$M = \cdots \to 0 \to \underset{\text{deg } 0}{M} \to 0 \to \cdots$$

$$H_n(\mathbf{R}\mathrm{Hom}(M,N)) = \mathrm{Ext}^{-n}(M,N)$$

$$X \overset{\mathbf{L}}{\otimes} Y = P \otimes Y$$

$$P \xrightarrow[\text{semi proj}]{\simeq} X$$

$\overset{\mathbf{L}}{\otimes}$ is associative: $(X \overset{\mathbf{L}}{\otimes} Y) \overset{\mathbf{L}}{\otimes} Z \simeq \cdots$, iso in $\mathcal{D}(R)$

Adjointness: $(\mathbf{R}\mathrm{Hom}(X \overset{\mathbf{L}}{\otimes} Y, Z) \simeq \mathbf{R}\mathrm{Hom}(Y, \mathbf{R}\mathrm{Hom}(X, Z)))$

Tensor evaluation

$$\mathbf{R}\mathrm{Hom}(X,Y) \overset{\mathbf{L}}{\otimes} Z \longrightarrow \mathbf{R}\mathrm{Hom}(X, Y \overset{\mathbf{L}}{\otimes} Z)$$

"Def": $\varphi \otimes z \longmapsto (x \longmapsto (-1)^{|x|\cdot|z|} \varphi(x) \otimes z)$

Tensor-evaluation is an iso if (e.g.)

- R Noeth, $H(X)$ bounded below, $H_\ell(X)$ f.g. for all ℓ, $H(Y)$ is bdd above, and $\underset{\text{flat dim}}{\mathrm{fd}_R} Z < \infty$ (i.e. $Z \simeq (0 \to F_s \to \cdots \to F_t \to$ flat mo

$(R, \mathfrak{m}, k)$ local (Commutative & Noetherian)

X R-complex, ~~depth~~

Def $\sup X = \sup\{n \mid H_n(X) \neq 0\} \in \mathbb{Z} \cup \{-\infty, \infty\}$

$\operatorname{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X)$

When $X = M$ module: $\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k, M)$
$= -\sup\{n \mid \mathrm{Ext}_R^{-n}(k, M) \neq 0\}$
$= \inf\{n \mid \mathrm{Ext}_R^{n}(k, M) \neq 0\}$

AB Formula (Iversen, –, Iyengar)
If X, Y are R-complexes, $H(X)$ bounded above, $\operatorname{fd}_R Y < \infty$,
then $\operatorname{depth}_R(X \otimes_R^{\mathbf{L}} Y) = \operatorname{depth}_R X - \sup\{k \otimes_R^{\mathbf{L}} Y\}$.

Very special case: $X = R$, $Y = N$ f.g. module with finite pd.
Then $\operatorname{pd}_R N = \operatorname{fd}_R N = \sup\{n \mid \mathrm{Tor}_n(k, N) \neq 0\} = \sup(k \otimes^{\mathbf{L}} N)$.

So the AB formula reads: $\operatorname{depth}_R N = \operatorname{depth} R - \operatorname{pd}_R N$
Auslander–Buchsbaum formula.

In AB formula, if we let $X = R$, we get
$$\operatorname{depth}_R Y = \operatorname{depth} R - \sup(k \otimes^{\mathbf{L}} Y)$$

Subtract from original formula, and we get

$$\operatorname{depth}_R(X \otimes^{\mathbf{L}} Y) = \operatorname{depth}_R X + \operatorname{depth}_R Y - \operatorname{depth} R,$$ assuming $\operatorname{fd}_R Y < \infty$

Foxby (4)

M, N modules s.t. $\mathrm{Tor}_i(M,N) = 0$ for $i > 0$:

$\mathrm{depth}\,(M \otimes_R N) = \mathrm{depth}_R M + \mathrm{depth}_R N - \mathrm{depth}\, R$, if $\mathrm{fd}\, N < \infty$.

{$\exists$ also sth. about CI-dim.}

<u>Def.</u> <u>Krull dim</u>

$$\dim X \stackrel{\mathrm{def}}{=} \cancel{\sup\{\dim R/p \mid p \in \mathrm{Spec}\, R}$$

$$\sup\{\dim(R/p) - \inf X_p \mid p \in \mathrm{Spec}\, R\}$$

$X = M$ module:

$$\dim M = \sup\{\dim R/p \mid p \in \mathrm{Supp}\, M\}$$

<u>Dimension Theorem</u> (–) If $H(X)$, $H(Y)$ finite (i.e. bounded, and f.g. in each degree), $H(Y) \neq 0$, and $\mathrm{fd}_R Y < \infty$. Then

($= \mathrm{pd}_R Y$)

$$\dim X \leq \dim(X \overset{L}{\otimes}_R Y) + \sup(k \overset{L}{\otimes}_R Y)$$

$$\cancel{\dim X = \mathrm{depth}\,(X \overset{L}{\otimes} Y) + \sup}$$

$$\mathrm{depth}\, X = \mathrm{depth}\,(X \overset{L}{\otimes} Y) + \sup(k \overset{L}{\otimes} Y)$$

subtract, and we get:

$$\dim X - \mathrm{depth}\, X \leq \dim(X \overset{L}{\otimes}_R Y) - \mathrm{depth}\,(X \overset{L}{\otimes}_R Y)$$

Special cases: (1) $X = M$, $Y = N$ both f.g., $\mathrm{pd}\, N < \infty$, $N \neq 0$.

then $\dim M \leq \dim(M \otimes_R N) + \mathrm{pd}_R N$. (Dim Thm.)

(2) $X = R$, $Y = F = (0 \to \overbrace{F_s \to \cdots \to F_0}^{\text{f.g. free}} \to 0)$, $H(F) \neq 0$, $H(F)$ finite length

Then $\dim R \leq s$ (New Intersection Theorem)

(Note: the dimension theorem is proved by reducing to the New Intersection Theorem.)

Amplitude Inequality (Iversen) Let X, Y be R-complexes s.t.

(1) $\operatorname{pd}_R Y < \infty$

(2) $H(Y) \neq 0$

(3) $H(Y)$ finite

(4) $H(X)$ finite.

Then $\sup X + \inf Y \leq \sup(X \overset{L}{\otimes} Y)$

By Nakayama, $\inf X + \inf Y = \inf(X \overset{L}{\otimes} Y)$.

Subtracting: $\operatorname{amp} X \overset{\text{def}}{=} \sup X - \inf X \leq \sup(X \overset{L}{\otimes} Y) - \inf(X \overset{L}{\otimes} Y) = \operatorname{amp}(X \overset{L}{\otimes} Y)$

$Y = N$ f.g., $\operatorname{pd} N < \infty$, $N \neq 0$, $r \in \mathfrak{m} \setminus \underset{\text{zerodivisors}}{Z(N)}$

$X = (0 \to R \xrightarrow{r} R \to 0)$

$X \otimes Y = (0 \to N \xrightarrow{r} N \to 0)$

Amp Ineq.: $r \notin Z(R)$.

Hence, $Z(R) \subseteq Z(N)$. ✓ (Zero-divisor theorem).

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Improved Version of Amp. Inequality (Iyengar & —)

Let X be an R-complex and Y an S-complex, S a local ring, where $R \longrightarrow S$ is a local homomorphism.

(1) $\mathrm{pd}_R Y < \infty$ (i.e. $\mathrm{fd}_R Y < \infty$)
(2) $H(Y) \neq 0$
(3) $H(Y)$ finite over S
(4) $H_\ell(X)$ finitely generated over R, for all ℓ.
($H(X)$ is not necc. bounded!)

Then $\sup X + \inf Y \leq \sup(X \otimes^{\mathbf{L}}_R Y)$

~~and~~ $\mathrm{amp}\, X \leq \mathrm{amp}\; X \otimes^{\mathbf{L}}_R Y$
(which now needs a proof)

Applications

Thm: Let $Q \to R \to S$ be local homomorphisms (of local rings) and let Y be an S-complex with $H(Y)$ finite, $H(Y) \neq 0$, and $\mathrm{fd}_R Y < \infty$. Then:

$$\mathrm{fd}_Q R + \inf Y \leq \mathrm{fd}_Q Y \leq \mathrm{fd}_Q R + \mathrm{fd}_R Y.$$

well-known.

proof: Let $k_Q = Q/m_Q$. $X \overset{\mathrm{def}}{=} k_Q \otimes^{\mathbf{L}}_Q R$ ($H(X)$ is not necc. bounded)

Then $\sup X = \mathrm{fd}_Q R$

$X \otimes^{\mathbf{L}}_R Y = k \otimes^{\mathbf{L}}_Q Y$

$\sup(X \otimes^{\mathbf{L}}_R Y) = \mathrm{fd}_Q Y$

So the result follows by the first inequality in Amp Inequality ✓

Cor: If $\mathrm{fd}_Q Y < \infty$, then $\mathrm{fd}_Q R < \infty$.

Consequence (w/ same assumptions on Y):

$$R \text{ regular} \Rightarrow Q \text{ regular}$$